

COMBINATORIAL TOPOLOGY OF THE STANDARD CHROMATIC SUBDIVISION AND WEAK SYMMETRY BREAKING FOR 6 PROCESSES

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ABSTRACT. In this paper we study a family of discrete configuration spaces, the so-called *protocol complexes*, which are of utmost importance in theoretical distributed computing. Specifically, we consider questions of the existence of compliant binary labelings on the vertices of iterated standard chromatic subdivisions of an n -simplex. The existence of such labelings is equivalent to the existence of distributed protocols solving Weak Symmetry Breaking task in the standard computational model.

As a part of our formal model, we introduce function $sb(n)$, defined for natural numbers n , called the *symmetry breaking* function. From the geometric point of view $sb(n)$ denotes the minimal number of iterations of the standard chromatic subdivision of an $(n - 1)$ -simplex, which is needed for the compliant binary labeling to exist. From the point of view of distributed computing, the function $sb(n)$ measures the minimal number of rounds in a protocol solving the Weak Symmetry Breaking task.

In addition to the development of combinatorial topology, which is applicable in a broader context, our main contribution is the proof of new bounds for the function $sb(n)$. Accordingly, the bulk of the paper is taken up by in-depth analysis of the structure of adjacency graph on the set of n -simplices in iterated standard chromatic subdivision of an n -simplex. On the algorithmic side, we provide the first distributed protocol solving Weak Symmetry Breaking task in the layered immediate snapshot computational model for some number of processes.

It is well known, that the smallest number of processes for which Weak Symmetry Breaking task is solvable is 6. Based on our analysis, we are able to find a very fast explicit protocol, solving the Weak Symmetry Breaking for 6 processes using only 3 rounds. Furthermore, we show that no protocol can solve Weak Symmetry Breaking in fewer than 2 rounds.

CONTENTS

1. Solvability of Weak Symmetry Breaking	2
1.1. Weak Symmetry Breaking as a standard distributed task	2
1.2. Previous work	3
1.3. Our results and outline of the paper	3
2. Combinatorics of iterated chromatic subdivisions	4
2.1. Standard chromatic subdivision of a simplex	4
2.2. Iterated chromatic subdivisions and the Weak Symmetry Breaking	5
2.3. The symmetry breaking function and its estimates	6
3. Combinatorial description of the simplicial structure of standard chromatic subdivisions	7

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3.1.	Combinatorics of partial ordered set partitions and the lower simplices in the standard chromatic subdivision	7
3.2.	Combinatorial language encoding simplices in iterated chromatic subdivisions	10
3.3.	Combinatorics of the pseudomanifold structure of the chromatic subdivision of a simplex	10
3.4.	Standard chromatic subdivisions and matchings	11
4.	The standard matching for the initial labeling	13
4.1.	Combinatorics of the second chromatic subdivision of an n -simplex	13
4.2.	Description of the initial labeling	13
4.3.	The Matching Lemma	14
4.4.	Using the Matching Lemma to get a partial matching	18
5.	Modifying the standard matching	18
5.1.	Conducting graphs	18
5.2.	Explicit description of 21 paths	27
6.	The distributed protocol solving Weak Symmetry Breaking for 6 processes in 3 rounds	29
7.	Weak Symmetry Breaking cannot be solved in 1 round	31
	References	32

1. SOLVABILITY OF WEAK SYMMETRY BREAKING

1.1. Weak Symmetry Breaking as a standard distributed task.

Let n be an integer, such that $n \geq 2$. The Weak Symmetry Breaking task for n processes is an inputless task, where the possible outputs are 0 and 1. A distributed protocol is said to solve the Weak Symmetry Breaking task if in any execution without failed processes, there exists at least one process which has value 0 as well as at least one process which has value 1.

In the classical setting, the processes know their id's, and are allowed to compare them. It is however not allowed that any other information about id's is used. The protocols with this property are called *comparison-based*.¹ In practice this means that behavior of each process only depends on the relative position of its id among the id's of the processes it witnesses and not on its actual numerical value. As a special case, we note that each process must output the same value in case he does not witness other processes at all. Weak Symmetry breaking is a standard task in theoretical distributed computing, and its solvability in the standard computational models is a sophisticated question which has been extensively studied.

For the rest of this paper we shall fix the computational model to be the layered immediate snapshot model, see [HKR]. In this model the processes use two atomic operations being performed on shared memory. These operations are: *write* into the register assigned to that process, and *snapshot read*, which reads entire memory in one atomic step. Furthermore, it is assumed that the executions are well-structured in the sense that they must satisfy the two following conditions. First, it is only allowed that at each time a group of processes gets active, these processes perform a write operation together, and then they perform a snapshot read operation together; no other interleaving in time of the write and

¹Alternative terminology *rank-symmetric* is also used in the literature.

read operations is permitted. Such executions are called *immediate snapshot* executions. Second, each execution can be broken up in rounds, where in every round each non-faulty process gets activated precisely once.

Even though this computational model is seemingly quite restrictive, it has been proved, see, e.g., [HKR], that most of the commonly used shared memory computational models are equivalent, in the sense of which of the tasks are solvable, to this one. As we will see, this model has the major advantage that the protocol complexes have a comparatively simple topological structure.

1.2. Previous work.

Several groups of researchers have studied the solvability of the Weak Symmetry Breaking by means of comparison-based layered immediate snapshot protocols. Due to primarily work of Castañeda and Rajsbaum, [CR08, CR10, CR12], it is known that the Weak Symmetry Breaking is solvable if and only if the number of processes is not a prime power; see also [AP12] for a counting-based argument for the impossibility part. This makes $n = 6$ the smallest number of processes for which this task is solvable.

The combinatorial structures arising in related questions on subdivisions of simplex paths have been studies in [ACHP, Ko13b]. The specific case $n = 6$ has been studied in [ACHP], who has proved the existence of the distributed protocol which solves the Weak Symmetry Breaking task in 17 rounds.

We refer to the classical textbook [AW], as well as the more recent monograph [AE14], for general background on theoretical distributed computing, and specifically on impossibility results. A general reference for topological methods in distributed computing is [HKR]. A general reference for combinatorial topology is [Ko07]. Finally, we recommend the survey [IRR11] for background on symmetry breaking tasks.

1.3. Our results and outline of the paper.

Our main mathematical result is the following theorem.

Theorem 1.1. *There exists a compliant binary labeling λ of the vertices of $\chi^3(\Delta^5)$, such that the restriction of λ to any 5-simplex of $\chi^3(\Delta^5)$ is surjective.*

This immediately implies the following theorem in theoretical distributed computing.

Theorem 1.2. *There exists a comparison-based layered immediate snapshot distributed protocol solving the Weak Symmetry Breaking task for 6 processes in 3 rounds.*

We also present the associated explicit distributed protocol. Because of the standard reduction to a mathematical question, see Theorem 2.8, we can derive Theorem 1.2 as a direct corollary of our main mathematical statement given in Theorem 1.1.

Our central method is the in-depth analysis of the simplicial structure of the second chromatic subdivision of a simplex. To this end, we need to develop the appropriate combinatorial language, in order to formalize and to work with standard chromatic subdivisions. This is done in Sections 2 and 4, yielding as a byproduct the combinatorial framework which is applicable in a much bigger generality.

Once the combinatorics is clear, the plan of our proof is as follows. First, in Section 5, we define some specially designed binary labeling on the vertices of the second chromatic subdivision of Δ^5 , calling this the *initial labeling*. This labeling is compliant, but it has many 1-monochromatic 5-simplices. In the next step, we find a perfect matching on the set of 1-monochromatic 5-simplices, such that whenever two 5-simplices are matched, they share a 4-simplex.

In Section 5 we describe the standard matching, which is easy to define, but which is not perfect. In Section 6 we modify this standard matching by connecting unmatched simplices by augmenting paths. This is a standard technique from matching theory, and it yields a perfect matching. Once we have a perfect matching, it is easy to use that to achieve a binary compliant labeling of the vertices of $\chi^3(\Delta^5)$ without any monochromatic simplices. This is done in the end of Section 6, and it proves our Main Theorem 1.1.

The remaining two sections are dealing with further specific issues. The explicit protocol for solving Weak Symmetry Breaking task for 6 processes in 3 rounds is given in Section 6. We finish with Section 7, where we give a very short argument for impossibility of solving Weak Symmetry Breaking task in 1 round for any number of processes.

While the focus of this paper has been on the case of 6 processes, it is possible to refine our analysis to a much more general setting, yielding new bounds for the symmetry breaking function, as well as producing distributed protocols for other numbers of processes. However, there is a number of technical issues which need to be resolved, and there is a number of new ideas which need to be introduced, since giving the simplex paths explicitly, as was done in Table 5.1, is prohibitive for higher values of n . For this reason, the extension of the techniques from this paper to higher values of n will appear in a separate article, see [Ko15, Ko16].

2. COMBINATORICS OF ITERATED CHROMATIC SUBDIVISIONS

2.1. Standard chromatic subdivision of a simplex.

For an arbitrary nonnegative integer n , we set $[n] := \{0, \dots, n\}$. We let Δ^n denote the standard n -simplex. Note that Δ^n has $n + 1$ vertices, which we index by the set $[n]$. For brevity, we shall skip the curly brackets for the sets consisting of a single element, and write expressions like $A \cup x$ and $A \setminus x$, rather than $A \cup \{x\}$ and $A \setminus \{x\}$.

Definition 2.1. *Given a set A and an element $x \in A$, we shall call the pair $\gamma = (A, x)$ a **node**. We shall say that x is the color of this node, and write $C(\gamma) = x$.*

When W is any set of nodes, we set $C(W) := \{C(\gamma) \mid \gamma \in W\}$. The reasons for our terminology will become apparent soon.

Definition 2.2. *An ordered set partition σ of a set A is an ordered tuple (A_1, \dots, A_t) of nonempty subsets of A such that A is a disjoint union of A_1, \dots, A_t . We shall use the notation $\sigma = (A_1 \mid \dots \mid A_t)$.*

Assume now that we are given an ordered set partition $\sigma = (A_1 \mid \dots \mid A_t)$ of a set A .

Definition 2.3. *We shall call the set $V(\sigma) := \{(A_1 \cup \dots \cup A_{i(x)}, x) \mid x \in A\}$ the **set of nodes** of σ , where $i(x)$ denotes the unique index such that $x \in A_{i(x)}$.*

Note that we have $V(\sigma) = \{(A_1 \cup \dots \cup A_k, x) \mid 1 \leq k \leq t, x \in A_k\}$.

Definition 2.4. *If (B, x) is a node of an ordered set partition σ of a set A , such that $|B| \geq |A| - 1$, then we say that x is **almost maximal** with respect to σ .*

Remark 2.5. *For every ordered set partition σ of a set A , there exist at least two elements of A , which are almost maximal with respect to σ .*

Indeed, if $\sigma = (A_1 \mid \dots \mid A_t)$, then all elements of A_t are almost maximal. If $|A_t| = 1$, then also all elements of A_{t-1} are almost maximal.

Given a permutation $\pi = (\pi_0 \dots \pi_n)$ of $[n]$, there is a natural ordered set partition of the set $[n]$ associated to it, namely $(\pi_0 | \dots | \pi_n)$; we shall call it $\hat{\pi}$. This way, we obtain precisely all ordered set partitions consisting of singletons.

There is a standard mathematical reformulation of the solvability of the Weak Symmetry Breaking task which we now proceed to describe. The central role in this description is played by the following abstract simplicial complex, called *standard chromatic subdivision of a simplex*.

Definition 2.6. *Let n be a nonnegative integer. The abstract simplicial complex $\chi(\Delta^n)$ is given as follows:*

- the set of vertices $V(\chi(\Delta^n))$ consists of all nodes (V, x) , such that $x \in V \subseteq [n]$;
- the maximal simplices of $\chi(\Delta^n)$ are indexed by ordered set partitions of $[n]$, where for each ordered set partition σ , its set of vertices is given by the corresponding set of nodes $V(\sigma)$;
- in general $S \subseteq V(\chi(\Delta^n))$ is a simplex if and only if there exists a maximal simplex σ such that $S \subseteq V(\sigma)$.

The color of the vertex of $\chi(\Delta^n)$ is the color of its indexing node, in particular, $C(-)$ can be applied to any subset of $V(\chi(\Delta^n))$. Furthermore, when σ is an n -simplex of $\chi(\Delta^n)$, we set $C(\sigma) := C(V(\sigma))$. Note that $\chi(\Delta^n)$ is a *pure* simplicial complex of dimension n , meaning that all of its maximal simplices have the same dimension.

The simplicial complex $\chi(\Delta^n)$ has been introduced by Herlihy & Shavit, [HS], see also a recent book [HKR], and is a widely used gadget in theoretical distributed computing. It has been proved in [Ko12] that $\chi(\Delta^n)$ is a subdivision of an n -simplex. A wide generalization of this fact has been proved in [Ko14a, Ko14b].

2.2. Iterated chromatic subdivisions and the Weak Symmetry Breaking.

The construction from Definition 2.6 can be used to define chromatic subdivision of an arbitrary simplicial complex due to the following simple fact: when restricted to any of its boundary simplices τ , the standard chromatic subdivision $\chi(\Delta^n)$ is naturally isomorphic to $\chi(\Delta^m)$, where m is the dimension of τ . Indeed, given any simplicial complex K , we can simply replace each of its simplices with its standard chromatic subdivision, these will fit together nicely, and we can call the result the *standard chromatic subdivision of K* . In particular, this means that we can define *iterated chromatic subdivisions* $\chi^k(\Delta^n)$. The simplicial complex $\chi^2(\Delta^2)$ is shown on Figure 2.1.

For any finite set A we let Δ^A denote the standard simplex whose vertices are indexed by the elements of A . In this setting, our standard simplex Δ^n would be called $\Delta^{[n]}$. The boundary simplices are called Δ^I , for all subsets $I \subset [n]$.

Given two equicardinal subsets $I, J \subset [n]$, the corresponding boundary simplices Δ^I and Δ^J have the same dimension, and any bijection $f : I \rightarrow J$ induces a simplicial isomorphism from Δ^I to Δ^J . Clearly, the construction of the iterated standard chromatic subdivision will also induce a simplicial isomorphism from $\chi^d(\Delta^I)$ to $\chi^d(\Delta^J)$. We let $\varphi_{I,J}$ denote the simplicial isomorphism from $\chi^d(\Delta^I)$ to $\chi^d(\Delta^J)$ induced by the unique order-preserving bijection from I to J .

Definition 2.7. *A binary labeling $\lambda : V(\chi^d(\Delta^n)) \rightarrow \{0, 1\}$ is called **compliant** if for all $I, J \subset [n]$, such that $|I| = |J|$, and all vertices $v \in V(\chi^d(\Delta^I))$, we have*

$$\lambda(\varphi_{I,J}(v)) = \lambda(v).$$

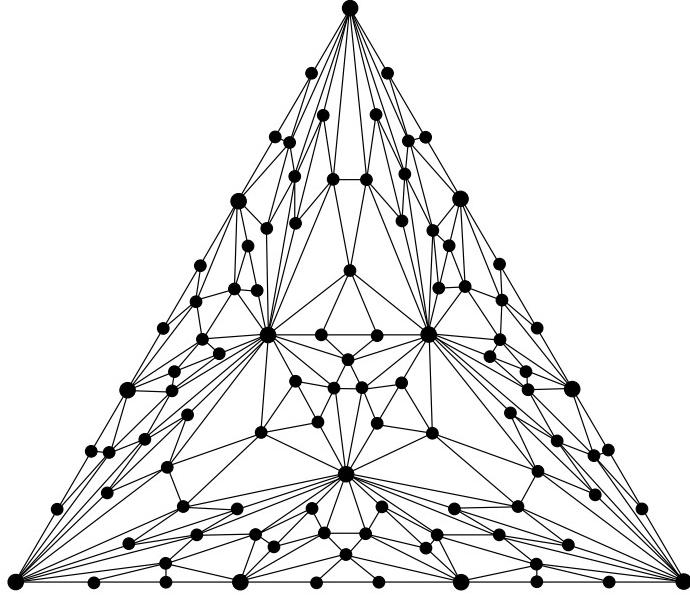


FIGURE 2.1. The second iterated standard chromatic subdivision of a 2-simplex.

In simple terms, this means that the restriction of the labeling to $\chi^d(\Delta^I)$ will only depend on the cardinality of the ordered set I .

The following well-known statement, see [ACHP, CR08, CR10, CR12, HKR, HS, Ko13b], is a useful reformulation of the solvability of the Weak Symmetry Breaking in the layered immediate snapshot model in purely mathematical terms.

Theorem 2.8. *In the layered immediate snapshot computational model, the Weak Symmetry Breaking task for n processes is solvable in d rounds if and only if there exists a compliant binary labeling $\lambda : V(\chi^d(\Delta^{n-1})) \rightarrow \{0, 1\}$, such that for every $(n-1)$ -simplex $\sigma \in \chi^d(\Delta^{n-1})$, the restricted map $\lambda : V(\sigma) \rightarrow \{0, 1\}$ is surjective.*

Clearly, the reduction in Theorem 2.8 means that Theorem 1.1 immediately implies Theorem 1.2.

2.3. The symmetry breaking function and its estimates.

We are now ready to introduce the main function for our study.

Definition 2.9. *Assume n is an arbitrary natural number. We let $sb(n)$ denote the minimal number d such that there exists a compliant binary labeling $\lambda : V(\chi^d(\Delta^{n-1})) \rightarrow \{0, 1\}$, such that for every $(n-1)$ -simplex $\sigma \in \chi^d(\Delta^{n-1})$, the restricted map $\lambda : V(\sigma) \rightarrow \{0, 1\}$ is surjective. If no such d exists, we set $sb(n) := \infty$.*

Note, that once such a labeling λ exists for some d , for any other $d' > d$ it is easy to extend it to a labeling $\lambda' : V(\chi^{d'}(\Delta^{n-1})) \rightarrow \{0, 1\}$ satisfying the same conditions.

Furthermore, note that Theorem 2.8 implies the following remark.

Remark 2.10. *The Weak Symmetry Breaking task for n processes is solvable in the layered immediate snapshot model if and only if $sb(n) \neq \infty$. The actual value $sb(n)$ is the minimal number of rounds needed for the distributed protocol to solve this task.*

The Table 2.1 summarizes our current knowledge of the function $sb(n)$.

Estimate	Source
$sb(n) = \infty$ if and only if n is a prime power	[CR08, CR10, CR12]
$sb(n) = O(n^{q+3})$, if n is not a prime power and q is the largest prime power in the prime factorization of n	[ACHP]
$sb(6) \leq 3$	Theorem 1.1
$sb(n) \geq 2$	Theorem 7.1

TABLE 2.1. The known estimates of $sb(n)$.

3. COMBINATORIAL DESCRIPTION OF THE SIMPLICIAL STRUCTURE OF STANDARD CHROMATIC SUBDIVISIONS

3.1. Combinatorics of partial ordered set partitions and the lower simplices in the standard chromatic subdivision.

Even though Definition 2.6 provides a well-defined simplicial complex, it is somewhat cumbersome to work with, mainly due to the fact that the lower-dimensional simplices lack a direct combinatorial description. We shall now mitigate this situation by giving an alternative combinatorial description of the simplicial structure of $\chi(\Delta^n)$, which has been first obtained in [Ko13a]. However, before we can do this, we need some additional terminology.

Definition 3.1. A **partial ordered set partition** of the set $[n]$ is a pair of ordered set partitions of nonempty subsets of $[n]$, $\sigma = ((A_1 | \dots | A_t), (B_1 | \dots | B_t))$, which have the same number of parts, such that for all $1 \leq i \leq t$, we have $B_i \subseteq A_i$. Given such a partial ordered set partition σ , we introduce the following terminology.

- The union $A_1 \cup \dots \cup A_t$ is called the **carrier set** of σ , and is denoted by $\text{carrier}(\sigma)$.
- The union $B_1 \cup \dots \cup B_t$ is called the **color set** of σ , and is denoted by $C(\sigma)$.
- The **dimension** of σ is defined to be $|C(\sigma)| - 1$, and is denoted $\dim \sigma$.

When appropriate, we shall also write

$$(3.1) \quad \sigma = \begin{array}{|c|c|c|} \hline A_1 & \dots & A_t \\ \hline B_1 & \dots & B_t \\ \hline \end{array},$$

which we shall call the *table form* of σ .

We note, that both nodes (A, x) , for $A \subseteq [n]$, as well as ordered set partitions of $[n]$ are special cases of partial ordered set partitions of $[n]$. Indeed, a node (A, x) , such that $A \subseteq [n]$ corresponds to the somewhat degenerate partial ordered set partition of $[n]$

$$\sigma = \begin{array}{|c|} \hline A \\ \hline x \\ \hline \end{array}$$

Whereas an ordered set partition $(A_1 | \dots | A_t)$ corresponds to the partial ordered set partition of $[n]$

$$\sigma = \begin{array}{|c|c|c|} \hline A_1 & \dots & A_t \\ \hline A_1 & \dots & A_t \\ \hline \end{array},$$

i.e., a partial ordered set partition $((A_1 | \dots | A_t), (B_1 | \dots | B_t))$, such that $A_i = B_i$ for all i , and $A_1 \cup \dots \cup A_t = [n]$.

Each partial ordered set partition has non-empty color set, which in turn is contained in its carrier set. The nodes correspond to the partial ordered set partitions with minimal

color set, consisting of just one element, and ordered set partitions correspond to the partial ordered set partitions with maximal color set, namely the whole set $[n]$.

Definition 3.2. Assume we are given a partial ordered set partition $\sigma = ((A_1 | \dots | A_t), (B_1 | \dots | B_t))$ of the set $[n]$, such that $\dim \sigma \geq 1$, and we are also given an element $x \in C(\sigma)$, say $x \in B_k$, for some $1 \leq k \leq t$. To define the **deletion of x from σ** we consider three different cases.

Case 1. If $|B_k| \geq 2$, then the deletion of x from σ is set to be

$$((A_1 | \dots | A_t), (B_1 | \dots | B_{k-1} | B_k \setminus x | B_{k+1} | \dots | B_t)).$$

Case 2. If $|B_k| = 1$, and $k \leq t-1$, then the deletion of x from σ is set to be

$$((A_1 | \dots | A_{k-1} | A_k \cup A_{k+1} | \dots | A_t), (B_1 | \dots | B_{k-1} | B_{k+1} | \dots | B_t)).$$

Case 3. If $|B_k| = 1$, and $k = t$, then the deletion of x from σ is set to be

$$((A_1 | \dots | A_{t-1}), (B_1 | \dots | B_{t-1})).$$

We denote the deletion of x from σ by $\text{dl}(\sigma, x)$.

It is easy to see that $\text{dl}(\sigma, x)$ is again a partial ordered set partition of $[n]$. We have $C(\text{dl}(\sigma, x)) = C(\sigma) \setminus x$, $\dim \text{dl}(\sigma, x) = \dim \sigma - 1$, and $\text{carrier}(\text{dl}(\sigma, x)) \subseteq \text{carrier}(\sigma)$.

Definition 3.2 can be generalized as follows.

Definition 3.3. Let $\sigma = ((A_1 | \dots | A_t), (B_1 | \dots | B_t))$ be a partial ordered set partition of the set $[n]$, and assume we are given a non-empty set $S \subset C(\sigma)$. Let $i_1 < \dots < i_m$ index all sets B_i , such that $B_i \not\subseteq S$, and set $i_0 := 0$. For all $1 \leq k \leq m$, we set

$$\tilde{A}_k := A_{i_{k-1}+1} \cup \dots \cup A_{i_k}, \text{ and } \tilde{B}_k := B_{i_k} \setminus S.$$

The obtained partial ordered set partition $((\tilde{A}_1 | \dots | \tilde{A}_m), (\tilde{B}_1 | \dots | \tilde{B}_m))$ is called **deletion of S from σ** , and is denoted $\text{dl}(\sigma, S)$.

The properties of the deletion of a part of its color set from a partial ordered set partition are summarized in the following proposition.

Proposition 3.4. Let $\sigma = ((A_1 | \dots | A_t), (B_1 | \dots | B_t))$ be a partial ordered set partition of the set $[n]$.

- (1) Assume we have a non-empty subset $S \subset C(\sigma)$. Then $\text{dl}(\sigma, S)$ is again a partial ordered set partition of the set $[n]$, such that
 - (a) $C(\text{dl}(\sigma, S)) = C(\sigma) \setminus S$;
 - (b) $\dim \text{dl}(\sigma, S) = \dim \sigma - |S|$;
 - (c) $\text{carrier}(\text{dl}(\sigma, S)) \subseteq \text{carrier}(\sigma)$.
- (2) Assume S and T are disjoint non-empty subsets of $\text{supp } \sigma$, such that $S \cup T \neq \text{supp } \sigma$. Then, we have

$$(3.2) \quad \text{dl}(\text{dl}(\sigma, S), T) = \text{dl}(\sigma, S \cup T).$$

Proof. All statements are immediate from Definition 3.3. □

Generalizing Definition 2.2 we can define the set of nodes of an arbitrary partial ordered set partition of the set $[n]$.

Definition 3.5. Let $\sigma = ((A_1 | \dots | A_t), (B_1 | \dots | B_t))$ be a partial ordered set partition of the set $[n]$. We shall call the set $V(\sigma) = \{(A_1 \cup \dots \cup A_{i(x)}, x) \mid x \in C(\sigma)\}$, the **set of nodes** of σ , where again $i(x)$ denotes the unique index such that $x \in B_{i(x)}$.

Note, that $V(\sigma) = \{(A_1 \cup \dots \cup A_k, x) \mid 1 \leq k \leq t, x \in B_k\}$, $|V(\sigma)| = \dim \sigma + 1$, and $C(V(\sigma)) = C(\sigma)$. Comparing Definitions 3.3 and 3.5 yields the identity

$$(3.3) \quad V(\sigma) = \{\text{dl}(\sigma, [n] \setminus \{x\}) \mid x \in C(\sigma)\}.$$

Furthermore, when an ordered set partition is viewed as a partial one, its set of nodes does not depend on which one of the Definitions 2.2 and 3.5 is used. Crucially, the set of nodes completely determines any partial ordered set partition of the set $[n]$.

Proposition 3.6. *Assume σ and τ are both partial ordered set partitions of the set $[n]$, such that $V(\sigma) = V(\tau)$, then $\sigma = \tau$.*

Proof. Assume $\sigma \neq \tau$. Without loss of generality, we can write $\sigma = ((A_1 \mid \dots \mid A_t), (B_1 \mid \dots \mid B_t))$ and $\tau = ((C_1 \mid \dots \mid C_q), (D_1 \mid \dots \mid D_q))$, such that $t \leq q$. To start with, the sets of sets $\{A_1, A_1 \cup A_2, \dots, A_1 \cup \dots \cup A_t\}$ and $\{C_1, C_1 \cup C_2, \dots, C_1 \cup \dots \cup C_q\}$ must be equal. This immediately implies that $t = q$, and that $A_i = C_i$, for all $i = 1, \dots, t$.

Since $C(\sigma) = C(\tau)$, we have $B_1 \cup \dots \cup B_t = D_1 \cup \dots \cup D_t$. Let k denote the minimal number such that $B_k \neq D_k$. Without loss of generality we can assume that we can find an element x , such that $x \in B_k$, $x \notin D_k$, and $x \in D_m$, for some $m > k$. Then, σ has a node $(A_1 \cup \dots \cup A_k, x)$, whereas τ has a node $(A_1 \cup \dots \cup A_m, x)$. Since $m \neq k$ we arrive at a contradiction. \square

We are now ready to prove that partial ordered set partitions provide the right combinatorial language to describe the simplicial structure of $\chi(\Delta^n)$.

Proposition 3.7. *The nonempty simplices of $\chi(\Delta^n)$ can be indexed by all partial ordered set partitions of $[n]$. This indexing satisfies the following properties:*

- (1) *The dimension of the simplex indexed by $\sigma = ((A_1 \mid \dots \mid A_t), (B_1 \mid \dots \mid B_t))$ is equal to $\dim \sigma$.*
- (2) *The vertices of the simplex σ indexed by $((A_1 \mid \dots \mid A_t), (B_1 \mid \dots \mid B_t))$ are indexed by $V(\sigma)$.*
- (3) *In general, the set of subsimplices of the simplex σ indexed by $((A_1 \mid \dots \mid A_t), (B_1 \mid \dots \mid B_t))$ is precisely the set of simplices indexed by partial ordered set partitions from the set $\{\text{dl}(\sigma, S) \mid S \subset C(\sigma)\}$.*

Proof. Assume $W \subseteq V(\chi(\Delta^n))$, such that W forms a simplex in $\chi(\Delta^n)$. By Definition 2.6 there exists an n -simplex σ of $\chi(\Delta^n)$, such that $W \subseteq V(\sigma)$. Furthermore, by (3.3) we get $W = \{\text{dl}(\sigma, [n] \setminus x) \mid x \in C(W)\}$. We now set

$$(3.4) \quad \tilde{\sigma} := \text{dl}(\sigma, [n] \setminus C(W))$$

to be the partial ordered set partition of the set $[n]$ which indexes the simplex W . First, we note that by Proposition 3.4(1)(a) we have

$$(3.5) \quad C(\tilde{\sigma}) = [n] \setminus ([n] \setminus C(W)) = C(W).$$

Furthermore, we derive

$$(3.6) \quad \begin{aligned} V(\tilde{\sigma}) &= \{\text{dl}(\tilde{\sigma}, C(\tilde{\sigma}) \setminus x) \mid x \in C(\tilde{\sigma})\} = \{\text{dl}(\tilde{\sigma}, C(W) \setminus x) \mid x \in C(W)\} = \\ &= \{\text{dl}(\text{dl}(\sigma, [n] \setminus C(W)), C(W) \setminus x) \mid x \in C(W)\} = \{\text{dl}(\sigma, [n] \setminus x) \mid x \in C(W)\} = W, \end{aligned}$$

where the penultimate equality is due to (3.2).

Assume τ is another n -simplex of $\chi(\Delta^n)$, such that $W \subseteq V(\tau)$. Set $\tilde{\tau} := \text{dl}(\tau, [n] \setminus C(W))$. We have shown in (3.6) that $V(\tilde{\sigma}) = V(\tilde{\tau}) = W$. It follows from Proposition 3.6 that $\tilde{\sigma} = \tilde{\tau}$, and hence the partial ordered set partition of the set $[n]$, which indexes W does not depend on the choice of σ .

We can now show that the indexing given by (3.4) satisfies the 3 properties in the formulation of the proposition. To see property (1), let $W \subseteq V(\chi(\Delta^n))$ be the simplex of $\chi(\Delta^n)$ indexed by $\tilde{\sigma}$. On one hand, $\dim W = |W| - 1$. On the other hand, picking an n -simplex σ such that $W \subseteq V(\sigma)$, we get

$$\dim(\text{dl}(\sigma, [n] \setminus C(W))) = \dim \sigma - (n + 1 - |C(W)|) = n - (n + 1 - |W|) = |W| - 1,$$

hence $\dim W = \dim \tilde{\sigma}$.

Finally note, that property (2) has been proved by (3.6), whereas property (3) follows directly from (3.2). \square

3.2. Combinatorial language encoding simplices in iterated chromatic subdivisions.

The following combinatorial concept is the key to describing the iterated standard chromatic subdivisions.

Definition 3.8. Assume we are given a tuple $\sigma = (\sigma_1, \dots, \sigma_d)$ of partial ordered set partitions of the set A . We say that the tuple σ is **linked** if for all $1 \leq i \leq d - 1$, we have $C(\sigma_i) = \text{carrier}(\sigma_{i+1})$.

Assume we have a partial ordered set partition $\sigma = ((A_1 \mid \dots \mid A_t), (B_1 \mid \dots \mid B_t))$, and $S \subset C(\sigma)$. Let i be the minimal index, such that $B_i \cup \dots \cup B_t \subseteq S$. We define $D(\sigma, S)$ to be $A_i \cup \dots \cup A_t$, if such an index i exists, and we let $D(\sigma, S)$ be empty otherwise. We can now generalize Proposition 3.7.

Proposition 3.9. The simplices of $\chi^d(\Delta^n)$ can be indexed by linked d -tuples of partial ordered set partitions of the set $[n]$. This indexing satisfies following properties:

- (1) The dimension of the simplex indexed by $\sigma = (\sigma_1, \dots, \sigma_t)$ is equal to $\dim \sigma_t$.
- (2) The subsimplices of $\sigma = (\sigma_1, \dots, \sigma_t)$ are all tuples $(\text{dl}(\sigma_1, S_1), \dots, \text{dl}(\sigma_t, S_t))$, where $S_t \subset C(\sigma_t)$, and $S_i = D(\sigma_{i+1}, S_{i+1})$, for all $1 \leq i \leq t - 1$.

Proof. Clearly, each simplex of $\chi^d(\Delta^n)$ can be obtained by first choosing a simplex of $\chi(\Delta^n)$, then viewing this simplex as Δ^k , for some $k \leq n$, then picking the next simplex in Δ^k , and so on, repeating d times in total. By Proposition 3.7 each next simplex can be indexed by a partial ordered set partition, and each time the color set of the previous simplex is the carrier of the next one. This is exactly the same as requiring for this tuple of simplices to be linked in the sense of Definition 3.8. Both (1) and (2) now follow immediately from Proposition 3.7 and the definition of the deletion operation. \square

Generalizing (3.1), the index of a simplex in $\chi^d(\Delta^n)$ can be visualized as an array of subsets:

A_1^1	\dots	$A_{t_1}^1$	\dots	A_1^d	\dots	$A_{t_d}^d$
B_1^1	\dots	$B_{t_1}^1$	\dots	B_1^d	\dots	$B_{t_d}^d$

satisfying $B_1^k \cup \dots \cup B_{t_k}^k = A_1^{k+1} \cup \dots \cup A_{t_{k+1}}^{k+1}$, for all $1 \leq k \leq d - 1$.

In particular, the top-dimensional simplices of $\chi^d(\Delta^n)$ are indexed by tuples $\alpha = ((A_1^1 \mid \dots \mid A_{t_1}^1), \dots, (A_1^d \mid \dots \mid A_{t_d}^d))$. We find it practical to use the following shorthand notation: $\alpha = (A_1^1 \mid \dots \mid A_{t_1}^1 \parallel \dots \parallel A_1^d \mid \dots \mid A_{t_d}^d)$.

3.3. Combinatorics of the pseudomanifold structure of the chromatic subdivision of a simplex.

One of the reasons, why the layered immediate snapshot computational model is amenable to detailed analysis is because the corresponding protocol complexes have a useful structure of a pseudomanifold.

Definition 3.10. A pure n -dimensional simplicial complex K is called a **pseudomanifold** of dimension n if the following two conditions are satisfied:

- (1) any $(n-1)$ -simplex belongs to at most two n -simplices;
- (2) any two n -simplices σ and τ can be connected by a path of n -simplices $\sigma = \sigma_0, \sigma_1, \dots, \sigma_t = \tau$, such that for any $k = 1, \dots, t$, the n -simplices σ_{k-1} and σ_k share an $(n-1)$ -simplex.

It is well known, see e.g., [HKR], that the iterated chromatic subdivision of an n -simplex is an n -pseudomanifold. We now provide the combinatorial language for describing how to move between n -simplices of this pseudomanifold.

To start with, consider an n -simplex σ of $\chi(\Delta^n)$, say $\sigma = (A_1 | \dots | A_t)$. We set

$$F(\sigma) := \begin{cases} [n] \setminus A_t, & \text{if } |A_t| = 1; \\ [n], & \text{otherwise.} \end{cases}$$

Definition 3.11. Given an n -simplex σ of the simplicial complex $\chi(\Delta^n)$, and $x \in F(\sigma)$, we let $\mathcal{F}(\sigma, x)$ denote the n -simplex of $\chi(\Delta^n)$ obtained in the following way. Let us say $\sigma = (A_1 | \dots | A_t)$, and $x \in A_k$.

Case 1. Assume $|A_k| \geq 2$, then we set

$$\mathcal{F}(\sigma, x) := (A_1 | \dots | A_{k-1} | x | A_k \setminus x | A_{k+1} | \dots | A_t).$$

Case 2. Assume $|A_k| = 1$ (that is $A_k = x$) and $k < t$. Then we set

$$\mathcal{F}(\sigma, x) := (A_1 | \dots | A_{k-1} | x \cup A_{k+1} | A_{k+2} | \dots | A_t).$$

Note that since $x \in F(\sigma)$, we cannot have the case $\sigma = (A_1 | \dots | A_{t-1} | x)$, hence $\mathcal{F}(\sigma, x)$ is well-defined. We say that $\mathcal{F}(\sigma, x)$ is obtained from σ by a *flip* of σ with respect to x . In this sense, $F(\sigma)$ is the set of all colors which can be flipped. Given any simplex σ we can always flip with respect to all colors except for at most one color. We can also flip back, so we have $\mathcal{F}(\mathcal{F}(\sigma, x), x) = \sigma$ for all σ, x .

Furthermore, we remark that for any $x \in F(\sigma)$, the n -simplices σ and $\mathcal{F}(\sigma, x)$ share an $(n-1)$ -simplex τ given by $\tau = \text{dl}(\sigma, x) = \text{dl}(\mathcal{F}(\sigma, x), x)$.

Definition 3.11 can be generalized to iterated chromatic subdivisions. For an n -simplex $\sigma = (\sigma_1 \parallel \dots \parallel \sigma_d)$ of $\chi^d(\Delta^n)$ we set $F(\sigma) := F(\sigma_1) \cup \dots \cup F(\sigma_d)$. Effectively this means that $F(\sigma) = [n]$, unless $F(\sigma_1) = \dots = F(\sigma_d) = [n] \setminus p$, in which case we have $F(\sigma) = [n] \setminus p$.

Definition 3.12. Assume we are given an n -simplex $\sigma = (\sigma_1 \parallel \dots \parallel \sigma_d)$ of the simplicial complex $\chi^d(\Delta^n)$, and $x \in F(\sigma)$. Let k be the maximal index such that $x \in F(\sigma_k)$, by the definition of $F(\sigma)$, such k must exist. We define $\mathcal{F}(\sigma, x)$ to be the following d -tuple of ordered partitions:

$$(3.7) \quad \mathcal{F}(\sigma, x) := (\sigma_1 \parallel \dots \parallel \sigma_{k-1} \parallel \mathcal{F}(\sigma_k, x) \parallel \sigma_{k+1} \parallel \dots \parallel \sigma_n).$$

Again, it is easy to see that for any $x \in F(\sigma)$, the n -simplices σ and $\mathcal{F}(\sigma, x)$ will share an $(n-1)$ -simplex, and that for all σ, x , we have the identity

$$(3.8) \quad \mathcal{F}(\mathcal{F}(\sigma, x), x) = \sigma.$$

3.4. Standard chromatic subdivisions and matchings.

Let us review some basic terminology of the graph theory, more specifically the matching theory. To start with, recall that a graph G is called *bipartite* if its set of vertices $V(G)$ can be represented as a disjoint union $A \cup B$, such that there are only edges of the type

(v, w) , with $v \in A$, $w \in B$. We shall call (A, B) a *bipartite decomposition*. Note, that such a decomposition need not be unique.

Classically, a *matching* in a graph G is a subset M of its set of edges $E(G)$, such that two different edges from M do not share vertices. We call edges which belong to M the *matching edges* and all other edges the *non-matching edges*. A vertex v is called *matched* if there exists an edge in M having v as an endpoint. We call the unmatched vertices *critical*, which is consistent with the terminology we used in the previous sections. A matching is called *perfect* if there are no critical vertices, otherwise we may call the matching *partial*. Clearly, an existence of a perfect matching implies that the sets A and B have the same cardinality. A matching is called *near-perfect* if there is exactly one critical vertex.

In this paper, the fundamental instance of a graph on which matchings are constructed is provided by Γ_n . This is the graph whose vertices are all n -simplices of $\chi(\Delta^n)$, and two vertices are connected by an edge if the corresponding n -simplices share an $(n-1)$ -simplex. Sometimes, we shall abuse our language and call the vertices of Γ_n simplices. We color the edges of Γ_n as follows: the edge connecting σ with τ gets the color of the vertex of σ which does not belong to $\sigma \cap \tau$. This graph has also been studied in [AC11]; an example is shown on Figure 3.1.

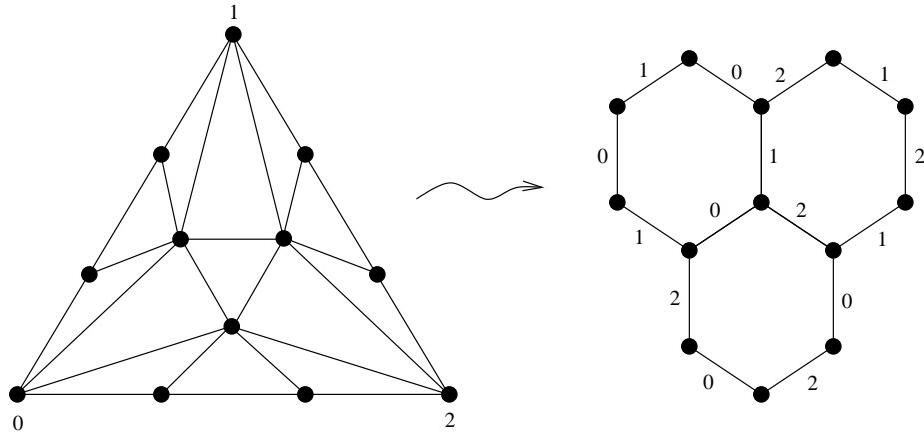


FIGURE 3.1. The standard chromatic subdivision of a 2-simplex and the corresponding graph Γ_2 , with labels showing the colors of the edges.

The graph Γ_n is bipartite and the bipartite decomposition is unique. Recall that n -simplices of $\chi(\Delta^n)$ are indexed by ordered set partitions $(A_1 | \dots | A_t)$ of the set $[n]$. The bipartite decomposition of Γ_n is then provided by sorting the n -simplices of $\chi(\Delta^n)$ into two groups according to the parity of the number t . For convenience of notations we shall define a function $O : V(\Gamma_n) \rightarrow \{\pm 1\}$, which we call *orientation* of the simplex, as follows:

$$O(A_1 | \dots | A_t) := \begin{cases} 1, & \text{if } t \text{ is even;} \\ -1, & \text{if } t \text{ is odd.} \end{cases}$$

The matching itself in this context will mean to group n -simplices in pairs, so that in each pair the n -simplices share a boundary $(n-1)$ -simplex. Accordingly, we can talk about critical n -simplices etc.

In general, for any $d \geq 1$, we let Γ_n^d denote the graph, whose vertices are the n -simplices of $\chi^d(\Delta^n)$, and two vertices are connected by an edge if the corresponding n -simplices share an $(n-1)$ -simplex. The edges of Γ_n^d are colored in the same way as those of Γ_n .

4. THE STANDARD MATCHING FOR THE INITIAL LABELING

4.1. Combinatorics of the second chromatic subdivision of an n -simplex.

In this paper we will primarily need the combinatorial description of the simplicial structure of the second chromatic subdivision of an n -simplex. In this case, the Proposition 3.9 says that simplices of $\chi^2(\Delta^n)$ are indexed by pairs $(\sigma \parallel \tau)$ of partial ordered partitions of $[n]$, such that $C(\sigma) = \text{carrier}(\tau)$. The n -simplices of $\chi^2(\Delta^n)$ are simply pairs $(\sigma \parallel \tau)$ of n -simplices of $\chi(\Delta^n)$. The vertices of $\chi^2(\Delta^n)$ are indexed by pairs of partial ordered set partitions

$$(4.1) \quad v = \begin{array}{|c|c|c||c|} \hline A_1 & \dots & A_t & S \\ \hline B_1 & \dots & B_t & x \\ \hline \end{array},$$

such that $S = B_1 \cup \dots \cup B_t$.

Assume we are given a simplex $\alpha = (\sigma \parallel \tau)$ of $\chi^2(\Delta^n)$, say $\sigma = ((A_1 | \dots | A_t), (B_1 | \dots | B_t))$, and $\tau = ((C_1 | \dots | C_q), (D_1 | \dots | D_q))$. Assume furthermore, we are given some subset $S \subset D_1 \cup \dots \cup D_q$. Let k be the minimal index such that $D_k \cup \dots \cup D_q \subseteq S$, then $\text{dl}(\alpha, S) = (\text{dl}(\sigma, C_k \cup \dots \cup C_q), \text{dl}(\tau, S))$.

Finally, we remark that for an n -simplex $\alpha = (A_1 | \dots | A_t \parallel B_1 | \dots | B_q)$, and $x \in [n]$, the flip $\mathcal{F}(\alpha, x)$ is always defined unless $A_t = B_q = x$, and is explicitly given by (3.7).

4.2. Description of the initial labeling.

For an arbitrary simplex $\sigma = (\sigma_1 | \dots | \sigma_d)$ of $\chi^d(\Delta^n)$ we set $\text{supp}(\sigma) := \text{carrier}(\sigma_1)$.

Definition 4.1. Let v be a vertex of $\chi^d(\Delta^n)$. We say that the vertex v is an **internal vertex** if $\text{supp}(v) = [n]$, otherwise we say that v is a **boundary vertex**.

In the case $d = 2$, there is a handy criterion for deciding whether all vertices of an n -simplex are internal.

Proposition 4.2. Let $\alpha = (A_1 | \dots | A_k \parallel B_1 | \dots | B_m)$ be an n -simplex of $\chi^2(\Delta^n)$. Then all vertices of α are internal if and only if $A_k \cap B_1 \neq \emptyset$.

In general, assume that $A_k \cap B_1 = \emptyset$, let q be the largest index such that $A_k \cap B_i = \emptyset$ for all $1 \leq i \leq q$. The boundary vertices of α are precisely the vertices with colors from $B_1 \cup \dots \cup B_q$.

Proof. Recall, that since α is an n -simplex, we have $A_1 \cup \dots \cup A_k = B_1 \cup \dots \cup B_m = [n]$. Pick $x \in [n]$, say $x \in B_l$. The vertex of α , which is colored by x has the index $(\tilde{\sigma} \parallel \tilde{\tau})$, where $\tilde{\sigma} = \text{dl}(\sigma, B_{l+1} \cup \dots \cup B_m)$, and $\tilde{\tau} = (B_1 \cup \dots \cup B_l | x)$. This vertex is internal if and only if $\text{carrier}(\tilde{\sigma}) = [n]$. By the definition of the deletion operation, this is the case if and only if $A_k \setminus (B_{l+1} \cup \dots \cup B_m) \neq \emptyset$, i.e.,

$$(4.2) \quad A_k \cap (B_1 \cup \dots \cup B_l) \neq \emptyset.$$

Clearly, the fact that (4.2) is true for all $l = 1, \dots, m$ is equivalent to the condition $A_k \cap B_1 \neq \emptyset$, and in general (4.2) yields the description of all boundary vertices of α . \square

We are now in a position to describe the initial labeling of the vertices of $\chi^2(\Delta^5)$

$$I : V(\chi^2(\Delta^5)) \longrightarrow \{0, 1\}.$$

Before we do this, we would like to designate certain boundary vertices of $\chi^2(\Delta^5)$ as exceptional.

Definition 4.3. The set \mathcal{V} consists of all vertices of $\chi^2(\Delta^5)$ listed in Table 4.1. The vertices in \mathcal{V} are called **exceptional vertices**, while all other boundary vertices are called **regular vertices**.

$\begin{array}{ c c } \hline a & a \\ \hline a & a \\ \hline \end{array}$	for all $0 \leq a \leq 5$;		
$\begin{array}{ c c } \hline a, b & a \\ \hline a & a \\ \hline \end{array}$	$\begin{array}{ c c } \hline a, b & a, b \\ \hline a, b & b \\ \hline \end{array}$	for all $0 \leq a < b \leq 5$;	
$\begin{array}{ c c } \hline a, b, c & a \\ \hline a & a \\ \hline \end{array}$	$\begin{array}{ c c } \hline a, b, c & a, b \\ \hline a, b & a \\ \hline \end{array}$	$\begin{array}{ c c } \hline a, b, c & a, b \\ \hline a, b & b \\ \hline \end{array}$	
$\begin{array}{ c c } \hline a, b, c & a, b, c \\ \hline a, b, c & b \\ \hline \end{array}$	$\begin{array}{ c c } \hline a, b, c & a, b, c \\ \hline a, b, c & c \\ \hline \end{array}$	for all $0 \leq a < b < c \leq 5$.	

TABLE 4.1. The 136 exceptional boundary vertices of $\chi^2(\Delta^5)$.

Clearly, there are 136 exceptional vertices. Note, that $|\text{supp } \sigma| \leq 3$ whenever v is an exceptional boundary vertex.

The general rule for the labeling I is now as follows:

$$I(v) := \begin{cases} 1, & \text{if } v \text{ is an internal vertex;} \\ 1, & \text{if } v \text{ is an exceptional boundary vertex;} \\ 0, & \text{if } v \text{ is a regular boundary vertex.} \end{cases}$$

Clearly, this labeling is compliant.

We remark that since any 5-simplex contains an internal vertex, there are no 0-monochromatic 5-simplices in the labeling I . Of course there are quite many 1-monochromatic 5-simplices. We will eliminate them by passing to the third chromatic subdivision.

4.3. The Matching Lemma.

Our next goal is to produce a perfect matching on the set of 1-monochromatic 5-simplices, so that each pair of matched 5-simplices shares a 4-simplex. We will start with producing a partial matching.

As a warm-up, let us as above consider the graph Γ_n , whose set of vertices is given by n -simplices of $\chi(\Delta^n)$. For $n \geq 1$, let f_n denote the number of vertices of Γ_n . We have $f_1 = 3$, $f_2 = 13$ etc. There is an easy recursion

$$(4.3) \quad f_n = \binom{n+1}{1} f_{n-1} + \binom{n+1}{2} f_{n-2} + \cdots + \binom{n+1}{n-1} f_1 + \binom{n+1}{n} + \binom{n+1}{n+1},$$

see [Ko14a, subsection 4.4] for a more general formula. One can see by induction that f_n is always odd. To do this, simply evaluate the right hand side of (4.3) modulo 2 and see that

$$f_n \equiv \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n-1} + \binom{n+1}{n} + \binom{n+1}{n+1} \equiv 2^{n+1} - 1 \equiv 1 \pmod{2}.$$

Since f_n is odd we cannot hope for a perfect matching. However, it is easy to produce a number of near-perfect matchings, each one matching all but one simplex. We need the following piece of terminology.

Definition 4.4. *Let $\Sigma = (x_0, \dots, x_k)$ be a fixed order on a nonempty subset of $[n]$, and let $\sigma = (A_1 | \dots | A_t)$ be an n -simplex of $\chi(\Delta^n)$. We shall define the number $\text{level}(\sigma, \Sigma) \in [n]$, called the **level** of σ with respect to Σ as follows. If $\sigma = (A_1 | \dots | A_{t-k-1} | x_0 | \dots | x_k)$, then $\text{level}(\sigma, \Sigma)$ is not defined; else $\text{level}(\sigma, \Sigma) := x_m$, where m is the highest number $0 \leq m \leq k$, such that $A_{t-k+m} \neq x_m$.*

Definition 4.4 can be rephrased as follows. If the last set in the ordered set partition σ is not x_k , the level $(\sigma, \Sigma) = x_k$, else we proceed with the next set in σ . If the last set in σ is x_k , but the penultimate one is not x_{k-1} , then level $(\sigma, \Sigma) = x_{k-1}$, else we proceed with the next set in σ . One then repeats this step, moving down in the ordered set partition σ . The level is then not defined, if we went through whole Σ and did not stop. This will happen precisely when σ has the form $(A_1 | \dots | A_{t-k-1} | x_0 | \dots | x_k)$.

We are now ready to define a special kind of matching, which we shall call the *standard matching*.

Definition 4.5. *Let $\Sigma = (x_0, \dots, x_n)$ be any fixed order on the set $[n]$. Recall that $\widehat{\Sigma}$ is an ordered set partition of $[n]$ associated to Σ . The **standard matching** M_Σ associated to Σ is a bijection*

$$M_\Sigma : \Gamma_n \setminus \widehat{\Sigma} \longrightarrow \Gamma_n \setminus \widehat{\Sigma},$$

defined by

$$\sigma \mapsto \mathcal{F}(\sigma, \text{level}(\sigma, \Sigma)).$$

There is a simple verbal formulation for the standard matching. In order to match a simplex σ , simply flip with respect to the highest possible color, such that the flip preserves the level function, see also the proof of Proposition 4.6.

The next proposition says that in the standard matching on the set of all n -simplices of $\chi(\Delta^n)$ with respect to any permutation Σ , there is exactly one critical simplex, namely $\widehat{\Sigma}$. This is illustrated by Figure 4.1.

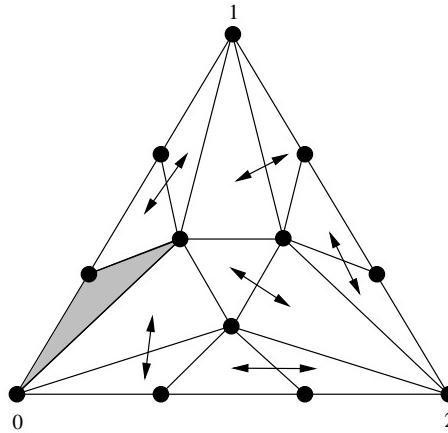


FIGURE 4.1. The standard matching on $\chi(\Delta^2)$ for $\Sigma = (0, 1, 2)$.

Proposition 4.6. *The standard matching M_Σ described in Definition 4.5 is a well-defined near-perfect matching on Γ_n , with the only critical n -simplex being indexed by $\widehat{\Sigma}$.*

Proof. Let $\sigma \in \Gamma_n \setminus \widehat{\Sigma}$. The crucial fact which we need is that

$$(4.4) \quad \text{level}(\sigma, \Sigma) = \text{level}(M_\Sigma(\sigma), \Sigma).$$

To see (4.4), assume $\sigma = (A_1 | \dots | A_m | x_{k+1} | \dots | x_n)$, such that $A_m \neq x_k$, i.e., $\text{level}(\sigma, \Sigma) = k$. The ordered set partition $M_\Sigma(\sigma)$ is obtained from σ by either merging two sets, or by splitting one of the sets. If in the process the sets A_m, x_{k+1}, \dots, x_n are untouched, then obviously $\text{level}(M_\Sigma(\sigma), \Sigma) = k$ as well. On the other hand, the only way the sets x_{k+1}, \dots, x_n could be modified would be by merging them with a neighbor. However, this is impossible, since merges must always involve the set x_k , and we have assumed that $A_m \neq x_k$.

It remains to consider the case that the set A_m was altered. If it was split into x_k and $A_m \setminus x_k$, then again (4.4) is satisfied, since $A_m \setminus x_k \neq x_k$. Finally, if the set A_m was merged with the set A_{m-1} , then again $A_{m-1} \cup A_m \neq x_k$, and (4.4) is valid again.

Now it follows from (4.4) together with (3.8) that

$$M_\Sigma(M_\Sigma(\sigma)) = \mathcal{F}(\mathcal{F}(\sigma, \text{level}(\sigma, \Sigma)), \text{level}(\sigma, \Sigma)) = \sigma.$$

This means that we have a well-defined matching, and that all simplices σ for which $\text{level}(\sigma, \Sigma)$ is defined are matched. We conclude that we have a near-perfect matching with the critical n -simplex being indexed by $\widehat{\Sigma}$. \square

Note, that even if we have $\Sigma = (x_0, \dots, x_m)$, for $m < n$, the matching M_Σ is still defined for those simplices σ , for which the number $\text{level}(\sigma, \Sigma)$ is defined. In this case, we will have a number of critical simplices, namely all those indexed by the tuples $(A_1 | \dots | A_k | x_0 | \dots | x_m)$, and the proof is simply the same as that of Proposition 4.6.

Definition 4.7. *For a proper nonempty subset $V \subset [n]$ we let $\Gamma_n(V)$ denote the subgraph of Γ_n induced by set of all n -simplices $\sigma = (A_1 | \dots | A_t)$, such that $A_1 \not\subseteq V$.*

Proposition 4.8. *Let $\Sigma = (x_0, \dots, x_k)$ be the set $[n] \setminus V$ taken in an arbitrary order. Then M_Σ provides a perfect matching of the simplices in $\Gamma_n(V)$.*

Proof. The part of the proof of Proposition 4.6 showing (4.4) did not use that fact that Σ has cardinality $n+1$, so we still have that identity. This means that M_Σ is still a well-defined matching on Γ_n , only now with many more critical simplices.

First, we note that $\text{level}(\sigma, \Sigma)$ is defined for all $\sigma \in \Gamma_n(V)$. Indeed, of $\text{level}(\sigma, \Sigma)$ is not defined, then $\sigma = (A_1 | \dots | A_k | x_0 | \dots | x_m)$, but then $A_1 \subseteq A_1 \cup \dots \cup A_k = [n] \setminus \{x_0, \dots, x_m\} = V$ yields a contradiction. Furthermore, we have the implication

$$\sigma \in \Gamma_n(V) \Rightarrow M_\Sigma(\sigma) \in \Gamma_n(V).$$

Indeed, assume $M_\Sigma(\sigma) = (B_1 | \dots | B_q)$, and $\sigma = (A_1 | \dots | A_t)$, with $A_1 \not\subseteq V$. Then, by definition of M_Σ , either we have $B_1 \supseteq A_1$, or $B_1 = x_k$, where $x_k = \text{level}(\sigma, \Sigma) \in [n] \setminus V$. Either way, we get $B_1 \not\subseteq V$, so $M_\Sigma(\sigma) \in \Gamma_n(V)$. We now conclude that M_Σ is a perfect matching on $\Gamma_n(V)$. \square

Definition 4.9. *Given an n -simplex $\sigma = (A_1 | \dots | A_t)$, with $A_t = \{x_1, \dots, x_k\}$, such that $x_1 < \dots < x_k$, we set $\text{St}(\sigma) := (x_1, \dots, x_k)$.*

Definition 4.10. *Given a subset $V \subset [n]$, we shall call a tuple $\mathcal{A} = (A_1 | \dots | A_k)$ of nonempty subsets of V a **prefix** in V , if A_1, \dots, A_k are disjoint. We shall say that a prefix is **full** if $A_1 \cup \dots \cup A_k = V$. We also allow an empty prefix.*

Furthermore, given a prefix $\mathcal{A} = (A_1 | \dots | A_k)$, we let $\Gamma_n(V, \mathcal{A})$ denote the subgraph of Γ_n induced by the set of all n -simplices $\sigma = (A_1 | \dots | A_k | A_{k+1} | \dots | A_t)$ of $\chi(\Delta^n)$, such that $A_{k+1} \not\subseteq V$.

In particular, the set $\Gamma_n(V, \emptyset)$ coincides with the previously defined set $\Gamma_n(V)$. We note, that for any $V \subset [n]$, the set of vertices of Γ_n , which is the set of all n -simplices of $\chi(\Delta^n)$, is a disjoint union of the sets of vertices of $\Gamma_n(V, \mathcal{A})$, where \mathcal{A} runs over all prefixes in V . The following lemma, whose statement is illustrated by Figure 4.2, provides a useful generalization of Proposition 4.8.

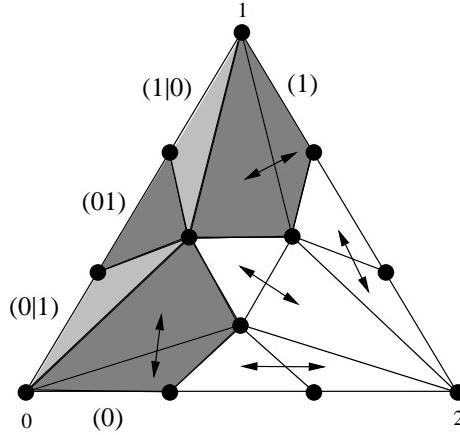


FIGURE 4.2. The decomposition of Γ_2 for $V = \{0, 1\}$.

Lemma 4.11. Assume $V \subset [n]$, and $\Sigma = (x_0, \dots, x_m)$ is an arbitrary permutation of the set $[n] \setminus V$. Let furthermore $\mathcal{A} = (A_1 | \dots | A_k)$ be an arbitrary prefix of V .

- (1) If the prefix \mathcal{A} is not full, then the standard matching M_Σ gives a perfect matching of simplices in $\Gamma_n(V, \mathcal{A})$.
- (2) If the prefix \mathcal{A} is full then the standard matching M_Σ of $\Gamma_n(V, \mathcal{A})$ has precisely one critical n -simplex, namely $\sigma = (A_1 | \dots | A_k | x_0 | \dots | x_m)$.

Proof. Our argument is very close to the proof of Proposition 4.8. Take $\sigma \in \Gamma_n(V, \mathcal{A})$. If $\text{level}(\sigma, \Sigma)$ is not defined, then σ has the form $(B_1 | \dots | B_t | x_0 | \dots | x_m)$. Since $\sigma \in \Gamma_n(V, \mathcal{A})$, we see that $\sigma = (A_1 | \dots | A_k | x_0 | \dots | x_m)$, which is only possible if the prefix \mathcal{A} is full. So, if \mathcal{A} is full, we do have one critical simplex, otherwise we do not.

Assume now $\sigma = (A_1 | \dots | A_k | A_{k+1} | \dots | A_t)$, such that $\sigma \neq (A_1 | \dots | A_k | x_0 | \dots | x_m)$, and let $x_t = \text{level}(\sigma, \Sigma)$. We have $A_{k+1} \cap V \neq \emptyset$. The ordered set partition $M_\Sigma(\sigma)$ is obtained from σ by merging two sets or splitting a set. If the sets A_1, \dots, A_{k+1} are untouched, then clearly $M_\Sigma(\sigma) \in \Gamma_n(V, \mathcal{A})$. In the other hand, the sets A_1, \dots, A_k cannot be involved in any modification, so the only remaining case is that A_{k+1} has been modified. Either A_{k+1} has been replaced by x_t followed by $A_{k+1} \setminus x_t$, or $A_{k+1} = x_t$ and we replace A_{k+1}, A_{k+2} with $x_t \cup A_{k+2}$. In any case, $x_t \in B_{k+1}$, where $M_\Sigma(\sigma) = (A_1 | \dots | A_k | B_{k+1} | \dots)$, hence $V \cup B_{k+1} \neq \emptyset$. Thus, we derive the implication

$$\sigma \in \Gamma_n(V, \mathcal{A}) \Rightarrow M_\Sigma(\sigma) \in \Gamma_n(V, \mathcal{A}),$$

for non-critical σ . Hence M_Σ gives a perfect matching on the set of all n -simplices in $\Gamma_n(V, \mathcal{A})$, with the only possible exception being $\sigma = (A_1 | \dots | A_k | x_0 | \dots | x_m)$. \square

4.4. Using the Matching Lemma to get a partial matching.

Before we proceed with describing the partial matching on the set of all 1-monochromatic 5-simplices of the labeling I , let us give combinatorial description of all the elements in this set. To start with, all 5-simplices with only internal vertices are 1-monochromatic. These are described by a simple criterion in Proposition 4.2. The rest of 1-monochromatic 5-simplices are those which contain internal vertices and some exceptional boundary vertices. In the Table 4.2 we list all exceptional boundary vertices and the 5-simplices which contain them.

Next we need to understand which of the 5-simplices from the right column of the Table 4.2 contain *only* exceptional boundary vertices. Recall that a 5-simplex of $\chi^2(\Delta^5)$ is indexed by a pair of ordered set partitions of subsets of [5]. We shall describe the set of 5-simplices containing only exceptional boundary vertices by grouping them together according to the first ordered set partition.

Definition 4.12. *For a 5-simplex σ of $\chi(\Delta^5)$, let $N(\sigma)$ denote the subgraph of Γ_n induced by all 5-simplices τ of $\chi(\Delta^5)$ such that the 5-simplex $(\sigma \parallel \tau)$ of $\chi^2(\Delta^5)$ is 1-monochromatic.*

The explicit description of the graph $N(\sigma)$ is given in Proposition 4.13, using Table 4.3. We remark that the types of the 5-simplices listed in the left column of that table are meant to be mutually exclusive.

Proposition 4.13. *Let $\sigma = (S_1 | \dots | S_p)$ be a 5-simplex of $\chi(\Delta^5)$, and set $V := [5] \setminus S_p$. Then we have $N(\sigma) = \Gamma_5(V) \cup \Lambda$, where the subset Λ is given as follows. If σ is of the type listed on the left column of Table 4.3, then*

$$(4.5) \quad \Lambda = \bigcup_{\mathcal{A}} \Gamma_5(V, \mathcal{A}),$$

where the union is taken over all prefixes \mathcal{A} which are listed in the corresponding row of Table 4.3. We set Λ to be an empty set otherwise.

Proof. To start with, we see that $V \subset [n]$. If V is empty, then we are looking at the central 5-simplex $\sigma = ([5])$, in which case all of the 5-simplices of $N(\sigma)$ are internal.

Assume now V is non-empty. Let $\tau = (T_1 | \dots | T_q)$ be a 5-simplex of $\chi(\Delta^5)$. By Proposition 4.2, all the vertices of the 5-simplex $(\sigma \parallel \tau)$ of $\chi^2(\Delta^5)$ are internal if and only if $S_p \cap T_1 \neq \emptyset$, or equivalently $T_1 \not\subseteq V$. In other words, the 5-simplex $(\sigma \parallel \tau)$ is internal if and only if $\tau \in \Gamma_5(V)$. All internal 5-simplices are 1-monochromatic, and all further 1-monochromatic 5-simplices appear due to the exceptional vertices. So if did not have any, we would simply have $N(\sigma) = \Gamma_5(V)$.

To take into account the influence of the exceptional vertices, we obtain the correction set Λ by a direct case-by-case analysis. Table 4.2 shows the cases which need to be considered. The Table 4.3 describes in each one of these cases, which new 1-monochromatic simplices will arise. \square

5. MODIFYING THE STANDARD MATCHING

5.1. Conducting graphs.

In order to look at matchings in depth, we need certain standard tools from the matching theory. Given a partial matching, there is a classical way of modifying it, possibly enlarging, or even making it perfect. Following definition provides the key concepts.

Definition 5.1. *Given a matching M on a graph G , assume γ is an edge path in G with endpoints v and w . We call the path γ*

exceptional vertices	5–simplices with modified labels
	$(a S_1 \dots S_p \parallel a T_1 \dots T_q)$
	$(a,b S_1 \dots S_p \parallel a T_1 \dots T_q)$ $(b a S_1 \dots S_p \parallel a T_1 \dots T_q)$
	$(a,b S_1 \dots S_p \parallel a,b T_1 \dots T_q)$ $(a,b S_1 \dots S_p \parallel a b T_1 \dots T_q)$
	$(a,b,c S_1 \dots S_p \parallel a T_1 \dots T_q)$ $(c a,b S_1 \dots S_p \parallel a T_1 \dots T_q)$ $(b a,c S_1 \dots S_p \parallel a T_1 \dots T_q)$ $(b,c a S_1 \dots S_p \parallel a T_1 \dots T_q)$ $(b c a S_1 \dots S_p \parallel a T_1 \dots T_q)$ $(c b a S_1 \dots S_p \parallel a T_1 \dots T_q)$
	$(a,b,c S_1 \dots S_p \parallel a,b T_1 \dots T_q)$ $(c a,b S_1 \dots S_p \parallel a,b T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel b a T_1 \dots T_q)$ $(c a,b S_1 \dots S_p \parallel b a T_1 \dots T_q)$
	$(a,b,c S_1 \dots S_p \parallel a,b T_1 \dots T_q)$ $(c a,b S_1 \dots S_p \parallel a,b T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel a b T_1 \dots T_q)$ $(c a,b S_1 \dots S_p \parallel a b T_1 \dots T_q)$
	$(a,b,c S_1 \dots S_p \parallel a,b,c T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel a,b c T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel b a c T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel a b c T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel b a,c T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel a b,c T_1 \dots T_q)$
	$(a,b,c S_1 \dots S_p \parallel a,b,c T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel a,c b T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel c a b T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel a c b T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel c a,b T_1 \dots T_q)$ $(a,b,c S_1 \dots S_p \parallel a b,c T_1 \dots T_q)$

TABLE 4.2. Exceptional boundary vertices and 5-simplices containing them; here we assume that $a < b < c$.

- **alternating** if the edges of γ are alternatively matching and non-matching;

type of σ	allowed prefixes
$(a S_1 \dots S_p)$	(a)
$(ab S_1 \dots S_p)$	$(a), (a b)$
$(abc S_1 \dots S_p)$	$(a), (a b), (ab), (ab c), (a bc), (a b c)$
$(a b S_1 \dots S_p)$	(a)
$(b a S_1 \dots S_p)$	$(a), (b)$
$(ab c S_1 \dots S_p)$	$(a), (a b)$
$(ac b S_1 \dots S_p)$	$(a), (a c)$
$(bc a S_1 \dots S_p)$	$(a), (b), (b c)$
$(a bc S_1 \dots S_p)$	(a)
$(b ac S_1 \dots S_p)$	$(a), (b)$
$(c ab S_1 \dots S_p)$	$(a), (c), (ab), (a b)$
$(a b c S_1 \dots S_p)$	(a)
$(a c b S_1 \dots S_p)$	(a)
$(b a c S_1 \dots S_p)$	$(a), (b)$
$(b c a S_1 \dots S_p)$	$(a), (b)$
$(c a b S_1 \dots S_p)$	$(a), (c)$
$(c b a S_1 \dots S_p)$	$(a), (b), (c)$

TABLE 4.3. New 1-monochromatic 5-simplices sorted by the supporting 5-simplex in $\chi(\Delta^5)$; here we assume $a < b < c$.

- **properly alternating** if it is alternating and those endpoints of the path which are not matched on the path are critical;
- **augmenting** if it is properly alternating, starting and ending with non-matching edges;
- **semi-augmenting** if it is properly alternating, starting with a matching edge and ending with a non-matching one;

- **weakly semi-augmenting** if it is alternating, starting with a matching edge and ending with a non-matching one;
- **non-augmenting** if it is alternating, starting and ending with matching edges.

Properly alternating non-self-intersecting edge paths allow us to modify matchings.

Definition 5.2. Assume we are given a matching M on a graph G , and a properly alternating non-self-intersecting edge path γ . We define $D(M, \gamma)$ be a new matching on G consisting of all edges from M which do not belong to γ together with all edges from γ which do not belong to M .

The next proposition shows that having a properly alternating edge path is enough to find a larger matching, and eventually to turn partial matchings into perfect ones.

Proposition 5.3. Assume M is a matching on a bipartite graph G , and v, w are critical vertices of G with respect to M . Assume furthermore, that there exists a properly alternating edge path γ from v to w . Then, there exists a matching \tilde{M} on G , such that the set of critical vertices with respect to \tilde{M} is obtained from the set of critical vertices with respect to M by removing the vertices v and w .

Proof. If the path γ is non-self-intersecting, then the new matching \tilde{M} can be taken to be $D(M, \gamma)$. Assume γ is a self-intersecting path, and its vertices, listed in the path order, are $v = v_0, v_1, \dots, v_l = w$.

Since γ is self-intersecting, there exist $0 \leq k < m \leq l$, such that $v_k = v_m$, and there is no vertex duplication in the sequence $v_k, v_{k+1}, \dots, v_{m-1}$. Let $\tilde{\gamma}$ be obtained from γ by removing the loop consisting of the edges $(v_k, v_{k+1}), (v_{k+1}, v_{k+2}), \dots, (v_{m-1}, v_m)$. Since the graph G is bipartite, that loop must have an even length, hence the new path $\tilde{\gamma}$ is again properly alternating. Repeating this procedure we will eventually arrive at a properly alternating non-self-intersecting edge path connecting v with w . \square

We note, that if the edge path γ in Proposition 5.3 is semi-augmenting, then the number of critical vertices does not change, whereas, if γ is non-augmenting, then the number of critical vertices increases by 2. Even though modifying a matching along a semi-augmenting path does not change the number of critical vertices, this modification is still useful as it could be thought of as transporting the critical vertex from one endpoint of the semi-augmenting path to the other.

Definition 5.4. Let G be a bipartite graph, with the corresponding bipartite decomposition (A, B) . We say that G is **conducting** if one of the following situations occur:

- (1) We have $|A| = |B| + 1$, and for any vertex $v \in A$, the graph G has a near-perfect matching, with the critical vertex v .
- (2) We have $|A| = |B|$, and for any vertices $v \in A, w \in B$, the graph G has a matching, with precisely two critical vertices v and w .

More specifically, if a conducting graph G satisfies condition (1) we shall call it **conducting graph of the first type**, else we shall call it **conducting graph of the second type**.

One handy way of showing that a graph is conducting is by presenting a certain collection of paths, as the next proposition details.

Proposition 5.5. Let G be a bipartite graph, with the corresponding bipartite decomposition (A, B) .

- (1) Assume M is a near-perfect matching of G with critical vertex $v \in A$, such that for any other vertex $w \in A$ there exists a semi-augmenting path from v to w , then the graph G is conducting of the first type.

- (2) Assume M is a perfect matching of G , such that for any vertices $v \in A, w \in B$ there exists a non-augmenting path from v to w , then the graph G is conducting of the second type.

Proof. This is an immediate consequence of Proposition 5.3. \square

Definition 5.6. Let Ω be a family of prefixes of a set $V \subset [n]$ containing the empty prefix. We say that Ω is **closed** if the following two conditions are satisfied:

- (1) if $(A_1 | \dots | A_k | x) \in \Omega$, where $A_1, \dots, A_k \subseteq V$, $x \in V$, then $(A_1 | \dots | A_k) \in \Omega$;
- (2) if $(A_1 | \dots | A_k) \in \Omega$, and $|A_m| \geq 2$, for some $1 \leq m \leq k$, then there exists $x \in A_m$, such that

$$(A_1 | \dots | A_{m-1} | x | A_m \setminus x | A_{m+1} | \dots | A_k) \in \Omega.$$

Furthermore, we let $\Gamma_n(V, \Omega)$ denote the subgraph of Γ_n induced by the vertices of $\bigcup_{\mathcal{A} \in \Omega} \Gamma_n(V, \mathcal{A})$.

Remark 5.7. By Lemma 4.11, the standard matching will induce a matching on $\Gamma_n(V, \Omega)$. However, this matching does not have to be perfect or near-perfect. As a matter of fact, the number of critical vertices will be equal to the number of full prefixes in Ω .

The next observation can be shown by direct inspection.

Remark 5.8. All families of prefixes Ω listed in Table 4.3 are closed.

We shall now proceed with analysing conductibility of graphs $\Gamma_n(V, \Omega)$. Let us fix a natural number $n \geq 3$. Let ρ denote the central simplex of $\chi(\Delta^n)$, indexed by $([n])$, and let ν denote the neighboring n -simplex indexed by $(n, [n-1])$. Consider the standard matching M_Σ , associated to $\Sigma = (0, 1, \dots, n)$. The only critical simplex of M_σ is $(0 | 1 | \dots | n)$, and clearly, M_σ matches ρ with ν . Our central tool will be the following technical lemma.

Lemma 5.9. Let σ be a non-critical n -simplex in $\chi(\Delta^n)$, for $n \geq 3$. There exists a weakly semi-augmenting (with respect to M_Σ) path P starting at σ and ending either at ρ or at ν .

Proof. Since the endpoints of a weakly semi-augmenting path have the same orientation, we know that if such a path P exists, then its endpoint from the set $\{\rho, \nu\}$ is uniquely determined. For brevity, we shall say that P goes from σ to $\{\rho, \nu\}$. Taking the empty path we see that the claim is true for $\sigma = \rho$ and for $\sigma = \nu$. Assume that $\sigma \neq \rho, \sigma \neq \nu$. We shall use the notation $\sigma = (A_1 | \dots | A_t)$ throughout the proof.

When describing edges in a path, we shall usually not list entire combinatorial indices of the simplices, but rather the part of the ordered set partition where the actual change occurs. The path from σ to $\{\rho, \nu\}$ will be constructed by concatenating different pieces.

Let us briefly introduce some notations in which we encode our paths. Assume $\sigma = (A_1 | \dots | A_t)$ is an n -simplex of $\chi(\Delta^n)$, and set $k := \text{level}(\sigma, \Sigma)$.

- We let \xrightarrow{k} denote the matching edge between σ and $M_\Sigma(\sigma) = \mathcal{F}(\sigma, k)$.
- Assume $A_i = \{x\}$, for some $1 \leq i \leq t-1$, and $x \neq k$. We let $\xrightarrow{x \cup}$ denote the edge between σ and $(A_1 | \dots | A_{i_1} | x \cup A_{i+1} | \dots | A_t)$.
- Assume instead that $|A_i| \geq 2$, for some $1 \leq i \leq t$, $x \in A_i$, and $x \neq k$. We let $\xrightarrow{x|}$ denote the edge between σ and $(A_1 | \dots | A_{i_1} | x | A_i \setminus x | \dots | A_t)$.
- If $|A_i| \geq 2$, for some $1 \leq i \leq t$, and $\text{level}(\sigma, \Sigma) \notin A_i$, we use the notation $\xrightarrow{A_i}$ to denote the set of edges $\xrightarrow{x|}$, for all $x \in A_i$.

We shall call the move \xrightarrow{A} the *generic split*. When a generic split is used in our notation of the path, it means that it does not matter at this point which of the elements of A we split off, and any will do. Formally, this means that we output a set of paths, rather than one path. Later, we will need to choose paths satisfying certain additional conditions. This can be done from this set of paths, by substituting specific splits instead of generic ones. For example, condition (2) of Definition 5.6 only guarantees the existence of *some* element x , which we are allowed to split. If at this point we used generic split of A_m in our path, then we will be able to pick a suitable path from our set.

We break our argument into considering three different cases.

Case 1. Assume that $\text{level}(\sigma, \Sigma) = n$. Pick i such that $n \in A_i$. Our assumption means that either $A_i = n$, and $i \leq t - 1$, or $|A_i| \geq 2$.

Our first goal, is to reduce this to the case $i = 1$. Assume that $i \geq 2$, and $|A_{i-1}| = 1$, say $A_{i-1} = a$. We can use the moves (5.1) and (5.2) to lower the index i by 1.

$$(5.1) \quad (\dots | a | n \cup B | \dots) \xrightarrow{n} (\dots | a | n | B | \dots) \xrightarrow{a \cup} (\dots | \{n, a\} | B | \dots)$$

$$(5.2) \quad (\dots | a | n | B | \dots) \xrightarrow{n} (\dots | a | n \cup B | \dots) \xrightarrow{a \cup} (\dots | \{n, a\} \cup B | \dots)$$

Assume now $i \geq 2$, and $|A_{i-1}| \geq 2$. We can use the moves (5.3) and (5.4) to lower the cardinality $|A_{i-1}|$ by 1, eventually reducing this to the case above.

$$(5.3) \quad (\dots | A_{i-1} | n \cup B | \dots) \xrightarrow{n} (\dots | A_{i-1} | n | B | \dots) \xrightarrow{A_{i-1}} (\dots | x | A_{i-1} \setminus x | n | B | \dots)$$

$$(5.4) \quad (\dots | A_{i-1} | n | B | \dots) \xrightarrow{n} (\dots | A_{i-1} | n \cup B | \dots) \xrightarrow{A_{i-1}} (\dots | x | A_{i-1} \setminus x | n \cup B | \dots)$$

In either case, we are able to lower the index i by 1, and in the end to reach the case $n \in A_1$.

Next, we want to achieve $|A_1| = 2$, or $|A_1| = |A_2| = 1$. Assume first $|A_1| \geq 3$, then use the move (5.5) to achieve $|A_1| = |A_2| = 1$.

$$(5.5) \quad (n \cup A | \dots) \xrightarrow{n} (n | A | \dots) \xrightarrow{A} (n | x | A \setminus x | \dots)$$

Assume now $|A_1| = 1$, but $|A_2| \geq 2$. Since $\sigma \neq \nu$, we must have $|A_3| \geq 1$. If $|A_3| \geq 2$, then use the move (5.6) to reduce to the case $|A_1| \geq 3$, which we just dealt with.

$$(5.6) \quad (n | A_2 | A_3 | \dots) \xrightarrow{n} (n \cup A_2 | A_3 | \dots) \xrightarrow{A_3} (n \cup A_2 | x | A_3 \setminus x | \dots)$$

If $|A_3| = 1$ and $A_4 \neq \emptyset$, then use the move (5.7) to reduce to the case $|A_1| \geq 3$ again.

$$(5.7) \quad (n | A_2 | b | A_4 | \dots) \xrightarrow{n} (n \cup A_2 | b | A_4 | \dots) \xrightarrow{b \cup} (n \cup A_2 | b \cup A_4 | \dots)$$

Finally, assume $|A_3| = 1, A_4 = \emptyset$. Then, use the move (5.8), picking $a \in \{n-1, n-2\} \cap A_2$, to reduce to the case $|A_1| = 2$.

$$(5.8) \quad (n | A_2 | b) \xrightarrow{n} (n \cup A_2 | b) \xrightarrow{a \downarrow} (a | (n \cup A_2) \setminus a | b) \\ \xrightarrow{n} (a | n | A_2 \setminus a | b) \xrightarrow{a \cup} (\{n, a\} | A_2 \setminus a | b)$$

So at this point we have $n \in A_1$, and $|A_1| = 2$, or $|A_1| = |A_2| = 1$.

Using one of the moves (5.9) and (5.10), whenever $|A_k| \geq 2$, we will eventually arrive at the simplex which is either encoded by $(n | a_1 | \dots | a_n)$ or by $(\{n, a_1\} | a_2 | \dots | a_n)$.

$$(5.9) \quad (n | a_1 | \dots | A_k | \dots) \xrightarrow{n} (\{n, a_1\} | \dots | A_k | \dots) \xrightarrow{A_k} (\{n, a_1\} | \dots | x | A_k \setminus x | \dots)$$

$$(5.10) \quad (\{n, a_1\} | \dots | A_k | \dots) \xrightarrow{n} (n | a_1 | \dots | A_k | \dots) \xrightarrow{A_k} (n | a_1 | \dots | x | A_k \setminus x | \dots)$$

We now continue with moves (5.11) and (5.12) reducing our simplex either to $(\{n, a\} | A)$ or to $(n | a | A)$.

$$(5.11) \quad (n | a_1 | \dots | a_t | A) \xrightarrow{n} (\{n, a_1\} | a_2 | \dots | a_t | A) \xrightarrow{a_t \cup} (\{n, a_1\} | a_2 | \dots | a_{t-1} | \{a_t\} \cup A)$$

$$(5.12) \quad (\{n, a_1\} | \dots | a_t | A) \xrightarrow{n} (n | a_1 | a_2 | \dots | a_t | A) \xrightarrow{a_t \cup} (n | a_1 | a_2 | \dots | a_{t-1} | \{a_t\} \cup A)$$

If $\sigma = (\{n, a\} | A)$, we use the move (5.13) to arrive at ν .

$$(5.13) \quad (\{n, a\} | A) \xrightarrow{n} (n | a | A) \xrightarrow{a \cup} (n | [n - 1])$$

So we can assume that $\sigma = (n | a | A)$. If $a = n - 1$, we use the move (5.14) to arrive at ρ .

$$(5.14) \quad (n | n - 1 | A) \xrightarrow{n} (\{n - 1, n\} | A) \xrightarrow{n-1} (n - 1 | n | A) \xrightarrow{n} (n - 1 | n \cup A) \xrightarrow{n-1 \cup} ([n])$$

Else write $A = n - 1 \cup B$ and use the move (5.15) to arrive at the considered case $a = n - 1$.

$$\begin{aligned} (5.15) \quad (n | a | A) &\xrightarrow{n} (\{n, a\} | A) \xrightarrow{n-1} (\{n, a\} | n - 1 | B) \xrightarrow{n} (n | a | n - 1 | B) \\ &\xrightarrow{a \cup} (n | \{n - 1, a\} | B) \xrightarrow{n} (\{n, n - 1, a\} | B) \xrightarrow{n-1} (n - 1 | \{n, a\} | B) \xrightarrow{n} (n - 1 | n | a | B) \\ &\xrightarrow{n-1 \cup} (\{n - 1, n\} | a | B) \xrightarrow{n} (n | n - 1 | a | B) \xrightarrow{a \cup} (n | n - 1 | a \cup B) \end{aligned}$$

Case 2. Assume level $(\sigma, \Sigma) \leq n - 2$. In this case we have $A_{t-1} = n - 1$ and $A_t = n$. We use the move (5.16) to reduce it to Case 1, where $k = \text{level}(\sigma, \Sigma)$.

$$(5.16) \quad (\beta | n - 1 | n) \xrightarrow{k} (\tilde{\beta} | n - 1 | n) \xrightarrow{n-1 \cup} (\tilde{\beta} | \{n - 1, n\})$$

Case 3. Assume $\text{level}(\sigma, \Sigma) = n - 1$. If $\sigma = (\dots | n - 1 \cup B | \dots | A | n)$, iteratively use the moves (5.17) and (5.18) to reduce it to Case 1.

$$\begin{aligned} (5.17) \quad (\dots | n - 1 \cup B | \dots | A | n) &\xrightarrow{n-1} (\dots | n - 1 | B | \dots | A | n) \\ &\xrightarrow{A} (\dots | n - 1 | B | \dots | x | A \setminus x | n), \text{ where } |A| \geq 2. \end{aligned}$$

$$\begin{aligned} (5.18) \quad (\dots | n - 1 \cup B | \dots | a | n) &\xrightarrow{n-1} (\dots | n - 1 | B | \dots | a | n) \\ &\xrightarrow{a \cup} (\dots | n - 1 | B | \dots | \{n, a\}) \end{aligned}$$

If $\sigma = (\dots | n - 1 | B | \dots | A | n)$, iteratively use the moves (5.19) and (5.20) to reduce it to Case 1.

$$\begin{aligned} (5.19) \quad (\dots | n - 1 | B | \dots | A | n) &\xrightarrow{n-1} (\dots | n - 1 \cup B | \dots | A | n) \\ &\xrightarrow{A} (\dots | n - 1 \cup B | \dots | x | A \setminus x | n), \text{ where } |A| \geq 2. \end{aligned}$$

$$\begin{aligned} (5.20) \quad (\dots | n - 1 | B | \dots | a | n) &\xrightarrow{n-1} (\dots | n - 1 \cup B | \dots | a | n) \\ &\xrightarrow{a \cup} (\dots | n - 1 \cup B | \dots | \{n, a\}) \end{aligned}$$

Assume now $\sigma = (\dots | n-1 \cup A | n)$. Use the moves (5.21) and (5.22) to reduce it to the previously considered cases.

$$(5.21) \quad (\dots | n-1 \cup A | n) \xrightarrow{n-1} (\dots | n-1 | A | n) \xrightarrow{A} (\dots | n-1 | x | A \setminus x | n), \text{ where } |A| \geq 2.$$

$$(5.22) \quad (\dots | \{n-1, a\} | n) \xrightarrow{n-1} (\dots | n-1 | a | n) \xrightarrow{a \cup} (\dots | n-1 | \{n, a\})$$

Finally assume $\sigma = (\dots | B | n-1 | A | n)$. We consider three cases. If $|B| \geq 2$, we use the move (5.23); if $|B| = 1$, we use the move (5.24); if $B = \emptyset$ use the move (5.25), which is possible since $n \geq 3$.

$$(5.23) \quad (\dots | B | n-1 | A | n) \xrightarrow{n-1} (\dots | B | n-1 \cup A | n) \xrightarrow{B} (\dots | x | B \setminus x | n-1 \cup A | n)$$

$$(5.24) \quad (\dots | b | n-1 | A | n) \xrightarrow{n-1} (\dots | b | n-1 \cup A | n) \xrightarrow{b \cup} (\dots | \{n-1, b\} \cup A | n)$$

$$(5.25) \quad (n-1 | [n-2] | n) \xrightarrow{n-1} ([n-1] | n) \xrightarrow{n-2} (n-2 | [n-3] \cup n-1 | n)$$

In each case we reduce to the already considered case $\sigma = (\dots | A \cup n-1 | n)$. \square

Remark 5.10. Curiously, Lemma 5.9 is not true when $n = 2$ for $\sigma = (1 | 0 | 2)$. It still remains true for all other simplices σ , and it is also trivially true for $n = 1$. All of this can be seen by direct inspection.

Theorem 5.11. For any natural number n , the graph Γ_n is conducting of the first type.

Proof. The cases $n = 1$ and $n = 2$ can be verified directly, so we assume $n \geq 3$. Recall now, that by Proposition 4.6 the standard matching M_Σ is a near-perfect matching with a critical n -simplex indexed by Σ . Let τ be any other n -simplex such that $O(\tau) = O(\Sigma)$. By Proposition 5.5(1) we need to show that there exists a weakly semi-augmenting path between τ and Σ . Let σ be any of the neighboring n -simplices of Σ . Since both σ and τ are non-critical, Lemma 5.9 implies that there exist a weakly semi-augmenting path Q_1 between τ and $\{\rho, \nu\}$, and a weakly semi-augmenting path Q_2 between σ and $\{\rho, \nu\}$. Since $O(\sigma) \neq O(\tau)$, the paths Q_1 and Q_2 will link them to the different vertices in $\{\rho, \nu\}$. Let Q be the concatenation of Q_1 , followed by the edge (ρ, ν) , then by Q_2 , and finally by the edge (σ, Σ) . Clearly, Q is a semi-augmenting path from τ to Σ . \square

Theorem 5.12. Assume $[n] \supset V \neq \emptyset$, such that $|V| \leq n-2$, then $\Gamma_n(V)$ is conducting of the second type.

Proof. Note, that the assumption $n \geq 3$ is automatic here, since $1 \leq |V| \leq n-2$. Without loss of generality, we can assume that $V = [k]$, with $0 \leq k \leq n-3$, and set $\Sigma := \{k+1, \dots, n\}$. By Proposition 4.8, the standard matching M_Σ is a perfect matching on the graph $\Gamma_n(V)$. Let ρ and ν be as above, we still have $\rho, \nu \in \Gamma_n(V)$ and $\nu := M_\Sigma(\rho)$. Pick an arbitrary $\sigma \in \Gamma_n(V)$, which is non-critical with respect to M_Σ . By Lemma 5.9 there exists a weakly semi-augmenting path Q from σ to $\{\rho, \nu\}$. All we need to do is to see that this path lies entirely within $\Gamma_n(V)$.

Recall, that there are 4 types of edges used in the construction of our path: \xrightarrow{k} , $\xrightarrow{x \cup}$, \rightsquigarrow , and \rightsquigarrow^{A_i} . Clearly, if $\sigma \in \Gamma_n(V)$, then any of the edges \xrightarrow{k} , $\xrightarrow{x \cup}$, \rightsquigarrow , \rightsquigarrow^{A_i} will lead to a vertex in $\Gamma_n(V)$ as well. The edges \rightsquigarrow^x occur only in paths (5.8), (5.14), (5.15), and (5.25). In all

these cases we have $x \in \{n-2, n-1\}$, in particular $x \notin V$; hence also here the edge $\xrightarrow{x|}$ leads to a vertex in $\Gamma_n(V)$.

We conclude that the entire path stays within $\Gamma_n(V)$. Now, we have seen in the proof of (1) that the existence of such paths means that any two vertices σ, τ of $\Gamma_n(V)$, such that $O(\sigma) = O(\tau)$, are connected by an augmenting path. By Proposition 5.5(2) this implies that $\Gamma_n(V)$ is conducting of the second type. \square

Theorem 5.13. *The graphs $\Gamma_5(V, \Omega)$ are conducting for all families of prefixes Ω listed in Table 4.3, under the assumption $|S_p| \geq 3$.*

Proof. Again without loss of generality we can assume that $n-2, n-1, n \in S_p$, all other cases can be reduced to this one by appropriate renaming. Choose Σ to be any order on $[n] \setminus V$ with $n-2, n-1, n$ coming last, i.e., $\Sigma = (\dots, n-2, n-1, n)$. By Remark 5.7 the standard matching M_Σ is a well-defined matching on the graph $\Gamma_n(V, \Omega)$ with one critical simplex for each full prefix in Ω . As above, we still have $\rho, \nu \in \Gamma_n(V)$ and $\nu := M_\Sigma(\rho)$.

Let σ be a non-critical simplex in $\Gamma_n(V, \Omega)$. By Lemma 5.9 there exists a weakly semi-augmenting path Q from σ to $\{\rho, \nu\}$. This time, what we need to do is to see that this path lies entirely within $\Gamma_n(V, \Omega)$. Again we need to perform a detailed analysis of moves (5.1) through (5.25).

Take any edge $e = (\sigma, \tau)$ on our path. Assume $\sigma \in \Gamma_n(V, \Omega)$, say $\sigma = (A_1, \dots, A_k, A_{k+1}, \dots, A_t)$, where $A_1, \dots, A_k \subseteq V$, and $A_{k+1} \not\subseteq V$. Of course we have $(A_1 | \dots | A_k) \in \Omega$. We need to show that $\tau \in \Gamma_n(V, \Omega)$. We consider separately each of the 4 possible types of edge e .

Case 1. If e is of the type \xrightarrow{k} , then the τ starts with the same prefix as σ , so the claim follows.

Case 2. Assume e is of the type $\xrightarrow{A_i}$. If $1 \leq i \leq k$, then, since Ω is a closed prefix, by Definition 5.6(2) there exists $x \in A_i$, such that the edge $\xrightarrow{x|}$ will lead to $\tau \in \Gamma_n(V, \Omega)$. If $k+2 \leq i \leq t$, then τ starts with the same prefix as σ , so again $\tau \in \Gamma_n(V, \Omega)$. Finally, the case $i = k+1$ never occurs on our path.

Case 3. Assume e is of the type $\xrightarrow{x|}$. As mentioned in the proof of Theorem 5.12, in all of these cases we have $x \in \{n-2, n-1\}$. This means that $x \notin V$, and τ starts with the same prefix as σ , so $\tau \in \Gamma_n(V, \Omega)$.

Case 4. Assume finally e is of the type $\xrightarrow{x\cup}$. We must have $\{x\} = A_i$, for some $1 \leq i \leq t$. If $i \geq k+1$, then τ starts with the same prefix as σ , so $\tau \in \Gamma_n(V, \Omega)$. If $i = k$, then τ starts with the prefix $(A_1 | \dots | A_{k-1})$. Since Ω is a closed prefix, Definition 5.6(1) implies that $(A_1 | \dots | A_{k-1}) \in \Omega$, hence $\tau \in \Gamma_n(V, \Omega)$. Finally, the case $i \leq k-1$ never occurs on our path.

Thus we have shown that $\tau \in \Gamma_n(V, \Omega)$ in all 4 cases. If Ω has no full prefixes, then by Lemma 4.11(1) $M(\Sigma)$ is a perfect matching and $\Gamma_n(V, \Omega)$ is conducting. If Ω has a single full prefix, then by Lemma 4.11(2) it has a near-perfect matching. Let γ denote the critical n -simplex. By an argument identical to the one used in the proof of Lemma 5.9, we see that there is a weakly semi-augmenting path from γ to any n -simplex σ of $\Gamma_n(V, \Omega)$, such that $O(\gamma) = O(\sigma)$. Using Proposition 5.5(1), we then conclude that $\Gamma_n(V, \Omega)$ is conducting.

Finally, when considering the row $(\{a, b, c\}, \{d, e, f\})$, with $a < b < c, d < e < f$, we get a family of prefixes with 3 full prefixes, hence we get 3 critical simplices. These simplices are $\tau_1 = (\{a, b\} | c | d | e | f)$, $\tau_2 = (a | b | c | d | e | f)$, $\tau_3 = (a | \{b, c\} | d | e | f)$. We now extend our matching M_Σ by matching τ_2 with τ_3 , obviously these share an $(n-1)$ -simplex, so the new matching is well-defined. All that remains is to find semi-augmenting paths from τ_2

and τ_3 to $\{\rho, \nu\}$. To find such a path starting from τ_2 we use the move (5.26), while to find such a path starting from τ_3 we use the move (5.27).

$$(5.26) \quad a|b|c|d|e|f \longrightarrow a|\{b,c\}|d|e|f \xrightarrow{e \cup} a|\{b,c\}|d|\{e,f\}$$

$$(5.27) \quad a|\{b,c\}|d|e|f \longrightarrow a|b|c|d|e|f \xrightarrow{e \cup} a|b|c|d|\{e,f\}$$

In each case, we arrive at a non-critical simplex, from which we have already found a weakly semi-augmenting path to $\{\rho, \nu\}$. \square

5.2. Explicit description of 21 paths.

Let us now return to the labeling I , where we want to find a perfect matching on the set of all 1-monochromatic 5-simplices. Using the notation of subsection 4.4, this set is a disjoint union of the sets $N(\sigma)$, where σ ranges over all 5-simplices of $\chi(\Delta^5)$. The complete description of $N(\sigma)$ has been obtained in Proposition 4.13.

At this point we would like to distinguish some of the 5-simplices of Γ_5 . We set

$$\begin{aligned} \mathcal{W}_1 &:= \{(a|[5] \setminus a) | a \in [5]\} \\ \mathcal{W}_2 &:= \{(\{a,b\}|[5] \setminus \{a,b\}) | a, b \in [5], a < b\} \\ \mathcal{W}_3 &:= \{(\{a,b,c\}|[5] \setminus \{a,b,c\}) | a, b, c \in [5], a < b < c\} \\ \mathcal{W} &:= \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \cup ([5]). \end{aligned}$$

Clearly, $|\mathcal{W}| = 42$, and we shall call the 5-simplices in \mathcal{W} *exceptional simplices*.

For every $\sigma \in \Gamma_5$, $\sigma = (A_1 | \dots | A_5)$, we now fix a matching on $N(\sigma)$. Specifically, we set $\Sigma := St(\sigma)$, see Definition 4.9, and let $\tilde{\mathcal{M}}_\sigma$ to be the standard matching \mathcal{M}_Σ , for all $\sigma \notin \mathcal{W}_3$. For $\sigma \in \mathcal{W}_3$, say $\sigma = (\{a,b,c\}, \{d,e,f\})$, with $a < b < c, d < e < f$, we obtain $\tilde{\mathcal{M}}_\sigma$ from the standard matching \mathcal{M}_Σ , by extending it by one more edge, matching $(a|b|c|d|e|f)$ with $(a|\{b,c\}|d|e|f)$, as in the proof of Theorem 5.13. We summarize what we know about these matchings in the following proposition.

Proposition 5.14. *The matching $\tilde{\mathcal{M}}_\sigma$ is perfect for all $\sigma \notin \mathcal{W}$, and near-perfect for all $\sigma \in \mathcal{W}$.*

Proof. If $\sigma = ([5])$, Proposition 4.6 implies that $\tilde{\mathcal{M}}_\sigma$ is near-perfect. Assume now $\sigma \neq ([5])$. By Proposition 4.8 the matching $\tilde{\mathcal{M}}_\sigma$ is perfect for all σ which do not appear in Table 4.3. Assume finally that σ does appear in Table 4.3. A line-by-line analysis shows that the corresponding prefix family Ω has full prefixes if and only if $\sigma \in \mathcal{W}$. It has one full prefix if $\sigma \in \mathcal{W}_1 \cup \mathcal{W}_2$, and it has three full prefixes if $\sigma \in \mathcal{W}_3$. The result now follows from Lemma 4.11, and the fact that two out of three critical (with respect to the standard matching) simplices for $\sigma \in \mathcal{W}_3$ have been matched in the extended matching $\tilde{\mathcal{M}}_\sigma$. \square

We let $\tilde{\mathcal{M}}$ denote the matching on Γ_5 obtained as a union of all matchings $\tilde{\mathcal{M}}_\sigma$. By Proposition 5.14 this matching has 42 critical simplices. We connect these 42 critical simplices by 21 non-intersecting augmenting paths; the Table 5.1 contains the explicit description of the 21 paths. This will allow to use the construction from Definition 5.2 and to deform the matching $\tilde{\mathcal{M}}$ to a perfect matching \mathcal{M} .

We define $R : \mathcal{W} \rightarrow \{\pm 1\}$, by setting $R(\sigma) := O(\tau)$, where $\tau \in \Gamma_5^2$ is the unique critical 5-simplex in $N(\sigma)$. Note that $R([5]) = -1$, $R(\sigma) = 1$ for $\sigma \in \mathcal{W}_1 \cup \mathcal{W}_2$, and $R(\sigma) = -1$ for $\sigma \in \mathcal{W}_3$. We now need the following result.

Lemma 5.15. *Assume that we have a path Q in Γ_5 connecting two exceptional 5-simplices σ and τ , such that*

012	→	0 12	→	0 1 2	→	0 1	→	01
013	→	0 13	→	0 3 1	→	0 3	→	03
014	→	0 14	→	0 4 1	→	0 4	→	04
015	→	0 15	→	0 5 1	→	0 5	→	05
023	→	0 23	→	0 2 3	→	0 2	→	02
024	→	4 02	→	4 2 0	→	4 2	→	24
025	→	2 05	→	2 5 0	→	2 5	→	25
034	→	3 04	→	3 4 0	→	3 4	→	34
035	→	3 05	→	3 5 0	→	3 5	→	35
045	→	4 05	→	4 5 0	→	4 5	→	45
123	→	1 23	→	1 2 3	→	1 2	→	12
124	→	1 24	→	1 4 2	→	1 4	→	14
125	→	1 25	→	1 5 2	→	1 5	→	15
134	→	3 14	→	3 1 4	→	3 1	→	13
234	→	2 34	→	2 3 4	→	2 3	→	23
135	→	1 35	→	1 3 5	→	1 3	→	1
145	→	4 15	→	4 1 5	→	4 1	→	4
235	→	3 25	→	3 2 5	→	3 2	→	3
245	→	2 45	→	2 4 5	→	2 4	→	2
345	→	5 34	→	5 3 4	→	5 3	→	5

$$0 \rightarrow 012345$$

TABLE 5.1. Explicit description of the 21 paths.

- (1) $R(\sigma) = -R(\tau)$;
- (2) any 5-simplex γ on the path Q other than σ and τ is non-exceptional, and the corresponding graph $N(\gamma)$ is conducting of the second type.

Then there exists an augmenting path T in Γ_5^2 such that

- (1) T connects the critical simplex in $N(\sigma)$ with the critical simplex in $N(\tau)$;
- (2) T lies entirely in $\bigcup_{\gamma \in Q} N(\gamma)$.

Proof. Assume $Q = (\gamma_1, \dots, \gamma_t)$, with $\gamma_1 = \sigma$, $\gamma_t = \tau$. For all $1 \leq i \leq t-1$, choose 5-simplices $\varphi_i \in N(\sigma_i)$, $\psi_i \in N(\sigma_{i+1})$ such that

- (1) φ_i and ψ_i are adjacent in Γ_5^2 ;
- (2) $O(\varphi_i) = R(\sigma)$.

To see that such simplices can be chosen let us pick α and β to be two adjacent 5-simplices in Γ_5 , say $\alpha = (A_1 | \dots | A_p)$, $\beta = (B_1 | \dots | B_q)$, and $\beta = \mathcal{F}(\alpha, x)$. Without loss of generality we can assume that $A_p \subseteq B_q$; in fact, we could always assume that either $A_p = B_q$, or $A_p \cup x = B_q$, but we do not need such detail. Pick any $y \in A_p$. Then any 5-simplex $(\alpha \parallel C_1 | \dots | C_r | x)$ in Γ_5^2 , such that $y \in C_1$ belongs to $N(\alpha)$. Furthermore, it is adjacent to the 5-simplex $(\beta \parallel C_1 | \dots | C_r | x)$, which in turn belongs to $N(\beta)$. There are 154 such 5-simplices, exactly half of which have orientation 1, so there is plenty of choice for the simplices φ_i and ψ_i .

Note that $O(\psi_i) = R(\tau)$ follows automatically from $O(\varphi_i) = -O(\psi_i)$ and $R(\sigma) = -R(\tau)$. Set ψ_0 to be the critical 5-simplex in $N(\sigma)$, and set φ_t to be the critical 5-simplex in $N(\tau)$.

By Theorem 5.13 there exist semi-augmenting paths Q_i , for $1 \leq i \leq t$, such that each Q_i connects ψ_{i-1} with φ_i . Concatenating these paths will yield the desired path T . \square

Since the paths presented in Table 5.1 are non-intersecting, it follows from Lemma 5.15 that the corresponding 21 augmenting paths in Γ_5^2 are non-intersecting as well. As we said above, this means that we can deform the matching $\tilde{\mathcal{M}}$ to a perfect matching \mathcal{M} , which leads to the proof of our main theorem. Before we proceed with the proof, we need one last piece of terminology.

Definition 5.16. Assume $v = (\sigma_1 \parallel \dots \parallel \sigma_d)$ is a vertex of $\chi^d(\Delta^n)$. The **predecessor** of v is the vertex of the simplex $\tau = (\sigma_1 \parallel \dots \parallel \sigma_{d-1})$ of $\chi^{d-1}(\Delta^n)$, which has the same color as v ; we shall denote it by $\text{Pred}(v)$.

Proof of Theorem 1.1. Let us describe an assignment L of values 0 and 1 to vertices of $\chi^3(\Delta^5)$. Let $w = (\sigma_1 \parallel \sigma_2 \parallel \sigma_3)$ be such a vertex, and set $\tau := (\sigma_1 \parallel \sigma_2)$, τ is a simplex of $\chi^2(\Delta^5)$. If $\dim \tau \leq 3$, then we set $L(w) := I(\text{Pred}(w))$. It is a compliant labeling, since the initial labeling I was chosen to be compliant. We shall call such the assignment $I(\text{Pred}(-))$ the *default value*.

Assume now $\dim \tau = 4$. If there is only one 5-simplex of $\chi^2(\Delta^5)$ containing τ , or if the two 5-simplices containing τ are not matched by \mathcal{M} , then we let $L(w)$ be the default value. Otherwise, we set $L(w) := 0$.

Finally, consider the case $\dim \tau = 5$. If τ is not a 1-monochromatic 5-simplex in I , then we let $L(w)$ be the default value. Otherwise, let $\mathcal{F}(\tau, c)$ be the 5-simplex which is matched to τ by \mathcal{M} . We set

$$(5.28) \quad L(w) := \begin{cases} 1, & \text{if } c = C(\sigma_3); \\ 0, & \text{otherwise.} \end{cases}$$

We shall now see that the obtained assignment L has no monochromatic 5-simplices. Let $\sigma = (\sigma_1 \parallel \sigma_2 \parallel \sigma_3)$ be a 5-simplex of $\chi^3(\Delta^5)$, and let $\tau = (\sigma_1 \parallel \sigma_2)$ be the corresponding 5-simplex of $\chi^2(\Delta^5)$ containing σ . If τ is not 1-monochromatic in I , then by our construction, all values of L on vertices of τ are the default ones, which means that the vertices of σ have the same values under L as vertices of τ of the same color under I . In particular, σ is not monochromatic.

Assume now τ is 1-monochromatic under I , and assume τ is matched to $\mathcal{F}(\tau, c)$ under \mathcal{M} . By our construction the vertex of σ which has color c is assigned the value 1, hence σ cannot be 0-monochromatic. Let $d \in [n]$, such that $c \neq d$, and d is almost maximal with respect to σ_3 ; such an element exists due to Remark 2.5. Let w be the color of σ with the color d . By our construction $L(w) = 0$, hence σ is not 1-monochromatic either. \square

6. THE DISTRIBUTED PROTOCOL SOLVING WEAK SYMMETRY BREAKING FOR 6 PROCESSES IN 3 ROUNDS

In this section we describe the application of our main theorem in theoretical distributed computing and present an explicit distributed protocol solving Weak Symmetry Breaking for 6 processes in 3 rounds. To start with, we record the information obtained in Section 5 in a table, which we call **ExSimp**. Namely, for each path from Table 5.1 use Lemma 5.15 to produce an augmenting path Q in Γ_5^2 . For every simplex $(\sigma \parallel \tau)$ from Q we add an entry (σ, τ, c) to **ExSimp**, where c is the color of the edge from Q which is adjacent to $(\sigma \parallel \tau)$, and which does not belong to original matching $\tilde{\mathcal{M}}$ (it does belong to the final matching \mathcal{M}).

```

protocol WSB6 (input:  $p$ )
write; read; write; read;
assume the current view is  $(\sigma' \parallel \tau')$ 
if  $\text{carrier}(\sigma') \neq [n]$  and  $(\sigma' \parallel \tau')$  is not in Table 4.1 then decide 0
write; read;
assume the current view is  $(\sigma \parallel \tau \parallel \gamma)$ 
if  $|\text{carrier}(\gamma)| \leq n - 1$  then decide 1
if  $\text{carrier}(\gamma) = [n]$  then  $\xi := \tau$ 
else assume  $\tau = (A_1 \mid \dots \mid A_t \parallel B_1 \mid \dots \mid B_t)$ , set  $q := [n] \setminus (B_1 \cup \dots \cup B_t)$ 
    if  $q \notin A_1 \cup \dots \cup A_t$  then decide 1
    else pick  $1 \leq k \leq t$  such that  $q \in A_k$ 
        set  $\xi := (A_1 \mid \dots \mid A_{k-1} \mid q \mid A_k \setminus q \mid A_{k+1} \mid \dots \mid A_t)$ 
if  $p = \text{Match}(\sigma, \xi)$  then decide 1
else decide 0

function Match (input:  $\xi, \sigma$ )
if  $(\xi, \sigma, c)$  is in ExSimp then return  $c$ 
else return  $\text{level}(\xi, \text{St}(\sigma))$ 

```

FIGURE 6.1. Protocol solving Weak Symmetry Breaking for 6 processes in 3 rounds.

The formal protocol is given in Figure 6.1. The verbal description is as follows. Let p be the id of the process running the protocol.

Step 1. Execute two rounds of write-read sequence. Assume that the view of the process p after 2 rounds is $v = (\sigma \parallel \tau)$. If $\text{carrier}(\sigma) = [n]$, then proceed to Step 2. Else we have $\text{carrier}(\sigma) \neq [n]$, in which case we check if v is one of the exceptional second round views from Definition 4.3, see Table 4.1. If it is an exceptional second round view from that table then proceed to Step 2, else **decide 0** and stop.

Step 2. Execute one more round of write-read. Assume that the view of the process p after 3 rounds is $(\sigma \parallel \tau \parallel \gamma)$.

- If $|\text{carrier}(\gamma)| \leq n - 1$, then **decide 1** and stop.
- If $\text{carrier}(\gamma) = [n]$, set $\xi := \tau$ and proceed to Step 3.
- If $|\text{carrier}(\gamma)| = n$, assume $\tau = (A_1 \mid \dots \mid A_t \parallel B_1 \mid \dots \mid B_t)$. We have $|B_1 \cup \dots \cup B_t| = n$, so there exists $q \in [n], q \neq p$, such that $B_1 \cup \dots \cup B_t = [n] \setminus q$. If $A_1 \cup \dots \cup A_t = [n] \setminus q$, then **decide 1** and stop. Else, there exists $1 \leq k \leq t$, such that $q \in A_k$. Now, set

$$\xi := (A_1 \mid \dots \mid A_{k-1} \mid q \mid A_k \setminus q \mid A_{k+1} \mid \dots \mid A_t).$$

Step 3. If $p = \text{Match}(\sigma, \xi)$, then **decide 1**, else **decide 0** and stop.

Here is the procedure **Match**, which has two ordered set partitions σ, ξ of $[n]$ as input and returns a value from $[n]$.

Procedure Match.

If (σ, ξ, c) is in **ExSimp**, set $\text{output} := c$, else set

$$\text{output} := \text{level}(\xi, \text{St}(\sigma)).$$

We note that the views from Table 4.1 can also be described verbally. For example, the view

a	a
a	a

means that process a has only seen itself both in the first and in the second round. The view

a, b, c	a, b
a, b	a

has a more lengthy interpretation. It means, that the process a has seen one other process b in the second round, whose id was larger than that of a . Furthermore, a has seen 3 processes in the first round, including itself and b . The third process it has seen has id larger than that of both a and b . Finally, the views of b and a in the first round were the same (a knows that after the second round). We leave it to the reader to provide similar interpretations for other views listed in the Table 4.1.

7. WEAK SYMMETRY BREAKING CANNOT BE SOLVED IN 1 ROUND

Currently, there are no lower bounds for the number of rounds needed to solve the Weak Symmetry Breaking task. Here we give the first such lower bound, stating that if Weak Symmetry Breaking can be solved at all, one would need at least two rounds to do that.

Theorem 7.1. *We have $sb(n) \geq 2$, for all $n \geq 2$. In other words, the Weak Symmetry Breaking task cannot be solved in one round for any value of $n \geq 2$.*

Proof. Let \mathcal{V}^n denote the set of pairs of integers $\{(r, p) \mid 1 \leq p \leq r \leq n\}$. Let us assume that we have a distributed protocol running exactly one round and having output in the set $\{0, 1\}$. After the execution of this protocol each process ρ has seen a certain number of other processes and has determined the relative position of its own id among the id's of the processes it has seen. Such an information can be encoded by a pair of integers (r, p) , such that $1 \leq r \leq n$, this is the number of processes, including itself, which ρ has seen, and $1 \leq p \leq r$, which is the relative position of the id of ρ . Hence, the decision function for such a protocol is precisely a function $\delta : \mathcal{V}^n \rightarrow \{0, 1\}$.

Assume now we are given such a decision function δ solving the Weak Symmetry Breaking. Let us break up $\mathcal{V}^n = \mathcal{V}_1^n \cup \dots \cup \mathcal{V}_n^n$, where $\mathcal{V}_i^n = \{(i, p) \mid 1 \leq p \leq i\}$. Let k be the maximum index such that $\delta_{\mathcal{V}_i^n}$ is not surjective. Since $\delta_{\mathcal{V}_1^n}$ is not surjective, the number k is well-defined. Without loss of generality we can assume that $\delta(k, 1) = \delta(k, 2) = \dots = \delta(k, k) = 0$. For all $k+1 \leq i \leq n$ we let α_i be any number such that $\delta(i, \alpha_i) = 0$. Such a number exists for all $k+1 \leq i \leq n$, since by our construction $\delta_{\mathcal{V}_i}$ is surjective for these values of i .

We shall now construct an execution E of the protocol after which the decision function will assign the value 0 to all processes, which of course contradicts to the claim that the protocol solves Weak Symmetry Breaking. The execution E will start with a simultaneous activation of k processes, after which the rest of the processes will activate one at a time. In total the execution E has $n - k + 1$ rounds.

More specifically, we can find numbers $\xi(1), \dots, \xi(n)$, such that the execution E is described as follows:

- (1) the id's of the processes participating in the first round are $\xi(1), \dots, \xi(k)$;
- (2) for each $r = 2, \dots, n - k + 1$, the id of the process participating in the round r is $\xi(k+r-1)$.

These numbers can be determined constructively as follows. To start with, the numbers $\xi(1), \dots, \xi(n)$ are initialized to be 0. We then proceed in $n - k + 1$ steps. In the first step, we set $\xi(i) := i$ for all $1 \leq i \leq k$. Next, for i running from $k + 1$ to n , repeat the following: set $\xi(i) := \alpha_i$, and increase by 1 each $\xi(j)$, such that $j < i$ and $\xi(j) \geq \alpha_i$. The result after $n - k + 1$ steps is then taken as the final output.

It is easy to see that each process activated in round r , for $2 \leq r \leq n - k + 1$, will see $k + r - 1$ processes, including itself, and that the relative position of his id in what he sees will be α_{k+r-1} . \square

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